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ABSTRACT

This proceedings of the annual conference of the International Group for the Psychology of Mathematics Education (PME) includes the following plenary papers and lectures: "Student Voice in Examining 'Splitting' as an Approach to Ratio, Proportions and Fractions" (J. Confrey); "Spontaneous and Scientific Concepts in Mathematics: A Vygotskian Approach" (V. John-Steiner); "Some Concerns about Bringing Everyday Mathematics to Mathematics Education" (A.D. Schliemann); and "Cognitive Growth in Elementary and Advanced Mathematical Thinking" (D. Tall). Panel Presentations include: "Video Protocols" (R. Nemirovsky); "Analysis of Classroom Interaction Discourse from a Vygotskian Perspective" (M.G.B. Bussi); "Mediation by Tools in the Mathematics Classroom" (L. Meira); and "Panel Contribution" (G. Vergnaud). Research fora topics are: "What is Algebraic Thinking?" and "How are Culture and Mathematical Knowledge Related?" Also included are 8 working group, 6 discussion group, 37 short oral, and 27 poster presentation abstracts as well as a listing of author addresses. (MKR)

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PME 19

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VOL. 1

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Universidade Federal de Pernambuco
Graduate Program in Cognitive Psychology

SE 057 135

Proceedings of the 19th PME Conference

Volume 1

INTERNATIONAL GROUP FOR
THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Proceedings of the 19th PME Conference

Volume 1

Recife, Brazil, July 22-27, 1995

Universidade Federal de Pernambuco
Graduate Program in Cognitive Psychology

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The cover painting reflects the integration of the PME Community with the warmth of the tropical sun and beaches of our home town and host site of PME 19: RECIFE.

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Preface

The program for PME 19 abounds in diversity. The Plenary Lectures, for example, focus on certain tensions—the opposition between spontaneous and scientific concepts, between everyday and school mathematics, elementary and advanced mathematical thinking, between student voice and oppression. The Panel Discussion provides three contrasting analyses of the same video footage. The Research Fora, each with reactors, highlight different perspectives on algebraic thinking and on the relations between culture and mathematical knowledge. And there are many excellent individual contributions, representing a variety of views on the Psychology of Mathematics Education.

We would like to think that the Brazilian climate, meteorological and social, the warmth of its people, the *ginga*, the *jeitinho*, the *abraços* and the musical rhythms can play a small yet important role in making PME 19 memorable. If not, we at least hope that you will find the following papers stimulating and engaging.

Recife, Brazil, May 4th, 1995
David Carraher & Luciano Meira

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PME came to being in 1976 at the Third International Congress on Mathematics Education (ICME3), held in Karlsruhe, Germany. Its past presidents have been Efraim Fischbein (Israel), Richard Skemp (UK), Gerard Vergnaud (France), Kevin Collis (Australia), Pearla Nesher (Israel), Nicolas Balacheff (France), and Kathleen Hart (UK). The major goals of the group are: (1) to promote International contacts and the exchange of scientific information in the Psychology of Mathematics Education; (2) to promote and stimulate interdisciplinary research in the aforesaid area with the cooperation of psychologists, mathematicians and mathematics teachers; and (3) to further a deeper and better understanding of the psychological aspects of teaching and learning mathematics and the implications thereof.

MEMBERSHIP

Membership is open to people involved in active research consistent with the aims of the Group, or professionally interested in the results of such research. Membership is on annual basis and depends on payment of the membership fees for the current year (January to December). For participants of the conference, the membership fee for the current year is included in the registration fee. Others are requested to write either to their Regional Contact, or directly to the PME Executive Secretary.

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PLENARIES

Student Voice in Examining "Splitting" as an Approach to Ratio, Proportions and Fractions[1]

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Interviewing students has been the basis of much of the research on student learning in mathematics education. In this paper, I suggest there are different ways of listening to student voice, and that the one which I have found most fruitful has required me to uncover and to challenge many widely-held and often tacit assumptions about mathematics. I will argue that such challenges are necessary to promote the kind of listening that will lead to all citizens becoming better educated and able to make critical decisions in the twenty-first century. If we wish to see a larger and more diverse group of students pursue mathematics, we, as mathematics educators, must criticize the forces which: 1) diminish the role of experience, tools, and physical demonstration in the pursuit of mathematics, 2) obscure the role of multiple forms of representation in discovery and proof, and 3) privilege certain forms of abstract symbolism, algorithms, and notation to the detriment of others in communication and learning. I will argue that mathematics educators have been far too timid in challenging the conduct and presentation of mathematics, and, as a result, mathematics education has been a primary vehicle of social and cultural reproduction in relation to racial, sexual and class inequities. The fact that in the vast majority of countries around the world, mathematics acts as a draconian filter to the pursuit of further technical and quantitative studies is held largely in place by our failures to recognize mathematics as a cultural force, to critique its impacts, and to offer more critical alternatives. I see the vigorous pursuit of student voice as a means to articulate a broader set of perspectives on mathematics and to open the field to a more diverse population.

This challenge, placed within the context of a meeting whose theme is the socio-cultural approaches to cognition, entails a basic denial that mathematics sits outside its constructions as human endeavor. It furthermore draws attention to any description of mathematics as acultural and ahistorical as itself a means of oppression. By denying the role of voice (the languages of human cultures in its construction), mathematics can remain a priesthood or a private club. By being acultural or ahistorical, it is cast as immune to critique of the composition of its workforce. Such a portrayal reinforces the view that mathematical talent surfaces, rather than being nurtured through effective schooling. We, in mathematics education, have an importantresponsibility to critique such a portrayal of mathematics, and to include student voice. I seek to make this critique, not by romanticizing the child, but, by showing

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[1] I wish to acknowledge the considerable contributions of Alan Maloney, David Dennis, Helen Doerr, and the mathematics education research group at Cornell University.

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with a particular example involving ratio and proportion, how listening to students can lead to the development of robust alternative points of view.

Articulating student voice is not simply a cognitive matter; it is not simply a socio-cultural issue of bringing children into alignment with adult expertise. Furthermore, voice, in this context refers to the feminist use of this term, not just as verbalizations--that which is said--but purposes, motivation, emotional presence, actions, feelings, activities, and expressions. "Voice" connotes fundamental matters of power: Whose voices are heard and given weight? When a group's voice is suppressed without physical threat, some form of schooling in self-silencing also occurs--the oppressed group has been given strong messages that certain forms of expression are either unacceptable or dangerous to themselves. Over time, their visible resistance declines and the messages to stay silent become internalized. They learn to participate in their own silencing as a means of survival. Those who do not learn these survival skills are either cast out of the system or labeled deviant. Both results have high emotional and intellectual costs.

If these forms of oppression exist in mathematics education, what would one expect to see? There would be changes as we move up the grades. Silences would increase. Certain questions, such as "what is this good for?" or "why are we learning this?" would be ignored. Certain forms of mathematics would be recognized and legitimated, while others would be cast aside, and on critical reflection, it would be difficult to decide why the alternative forms were disregarded. There would be large reductions in participation, and those reductions would tend to eliminate members of certain subgroups--by race, by gender, by class, by cognitive, social or emotional preference. The fact that so many voices become reduced to a few would not be grounds for outrage, but viewed as a regrettable but necessary part of intellectual development. Attempts to remove these forms of oppression would be time-consuming, would encounter barriers and resistance both from those oppressed and the oppressors, and, when successful, would lead to the release of dramatic expressions of debilitating and facilitating emotions, painful memories, self doubt, and exhilarating insights.

Theoretical Foundations

My own work has progressed through a variety of theoretical positions and none alone has been sufficient to produce these claims. Radical constructivism and philosophies of science allowed me to examine the development of ideas, and engage in historical analysis. These theories initiated me into relativism and set the problem of how to undertake scholarship within an admission of the inevitable fallibility of human thought (Ernest, 1991; von Glasersfeld, 1995). Radical constructivism also presented the duality of the seer and the seen and stubbornly refused to allow objectivity to mask the role of the subject. Socio-cultural perspective, and to some degree, social constructivism made it evident that strong cultural tides envelop all forms of knowledge development. In particular, Vygotskian theory encouraged me to see the interplay between one's experience and one's social interactions, between tools and language (Confrey, 1993, 1994a, 1994b, 1995).

A weakness in the Vygotskian socio-cultural theory was the absence of a way in which children's inventions could lead to changes in adult reasoning. There appeared to be no means of radical criticism in Vygotskian socio-cultural perspective. Critical theory in mathematics made solid critiques of the mathematics, but tended to stop at the use statistics and arithmetic (Frankenstein, 1989). Experience in mothering, living in wonder of children's perceptions, together with Piagetian developmental theory, secured my commitment to the richness and legitimacy of children's voices. Finally, the stark reality of a white male dominance of mathematics departments and the research academy in the United States made it clear that a more critical examination of the whole enterprise needed to be undertaken.

Feminist theory has been relatively modest in its examination of mathematics, sometimes even endorsing its freedom from critical analysis, sometimes offering wistful critiques of the content. Making a feminist critique of mathematics (Jacobs, 1994; Burton, 1992; Rogers, 1990; Buerk, in press) has been difficult, yet the tools for doing so are becoming richer and more convincing. I found feminist analysis to be useful for a number of reasons: Firstly, as Marilyn Frye (1983) described it, oppression is like a bird cage: look at any single wire and you wonder why the bird doesn't just fly around it. Look at the configuration of the whole and the entrapment becomes evident. Secondly, feminist theory makes the issue of emotional well-being a fundamental consideration: an emotionally impoverished environment can be an understandable source of alienation, and one's failure to thrive in such an environment no longer is simply evidence of a person's lack of hardiness or determination. Thirdly, feminism begins with the analysis of participation structures, so that one examines the impact on groups. In constructivism, and in most non-feminist developmental theories, the construction and analysis of the impact on groups is an awkward endeavor.

What has yet to be undertaken to my satisfaction is a feminist epistemological critique of particular content, organization, and structure in mathematics. Feminist critiques in science and elsewhere could point the way (Harding, 1986; Bbeier, 1984). Feminism's commitment to challenging bias and elitism, especially as related to hierarchical structures, points towards examining the role of hierachies in mathematical thought . At the same time as I sought to critically examine the concept of abstraction in mathematics, feminist theory was examining the ways in which disembodiment of knowledge tends to disadvantage or threaten females and children. Using these pointers, I had to apply them to the particular subject matter of mathematics, a subject matter assumed by most to be pure and independent from human intervention. This made the undertaking relatively original work--forging a theory of mathematical development, revising current convention, while working with the current artifacts. The theory was ripe for such an application--and so, weaving between radical constructivism, socio-cultural perspective and feminism, I have made a direct attempt.

In effect, I am suggesting that multiple theories were necessary to offer the critique of the imperatives that promote silence in mathematics. I am further suggesting that feminism provides the perspectives and tools that allowed me to make the fundamental step--to take the side of the children's voices against the dominant views in mathematics.

No single analysis can ever make the case for the oppression of children's voices. However, the corpus of my work and that of others (Mokros & Russell, 1992; Maher, 1994; Duckworth, 1987; Ackermann, 1991; Kamii, 1985; Nemirovsky, 1993; Monk, 1992) has established the viability of the approach. Over the years, as I learned to listen more and more closely to students' inventions and perceptions, a bred-in-the-bone conviction emerged. In those children's voices sounded more ingenuity, more reason, more conviction, and more critical challenge to current practice in mathematics than most of us were prepared (or willing) to hear. These investigations led me into philosophy and history as I sought to strengthen my ability to hear and validate the nascent wisdom in those voices. Slowly, gradually, I began to question the (so-called constructivist) practices[2] that invited students to speak, only to find that their ideas were not being heard--at one end of the spectrum by those whose mathematics was too weak to permit listening, and at the other end by those whose mathematical training was too effective to permit listening.

As I and others struggled to hear and then tell these stories of children's words, our own voices met resistance. Some educators found the stories quaint, but argued for the need to quickly bring the children back on track, a track which, of course, they could identify with confidence, their success and training giving them the necessary qualifications to decide. Others found the stories quaint, but argued that they were irrelevant relative to the demands of whole classes. Others found the stories quaint, and made the telling of stories the curriculum itself, leaving the children with their words and methods, but with little to grow on. In each of these three cases, the children were underserved, and the hard questions of what constitutes educational progress in mathematics were being inadequately answered. Our own limited views of the content were restricting children's proposals. And the result was that: 1) children found mathematics to be nonsensical; 2) they found out they didn't need to take initiative, or worse, that initiative produced negative effects; and 3) on the whole, the social stratifications in class, race and gender based on the culture, parental, teacher, peer, or self expectations were replicated in succeeding generations.

What I am ultimately claiming is that unless mathematics educators challenge the dominance of the curriculum by particular brands and values of pure mathematics , we will fail to set an aggressive enough equity agenda. I am further saying the challenge I will lay out is difficult, for it requires us to challenge our own training and to question deeply our own tacit commitments and convictions. It also requires us to forge new partnerships, forego some of our past loyalties to pure or abstract mathematics, and to recruit different sorts of people into our field. In this paper I seek to demonstrate that student voice is an effective and accessible vehicle for making these challenges.

To make this case, I will begin by introducing the idea of a voice-perspective dialectic. By voice, I refer to students' expressions of their ways of thinking, doing and describing, both alone and in groups. I have come to believe that voice can only be strengthened if one acknowledges that its articulation is placed within a voice-perspective dialectic. Perspective refers to the listener's frame of reference as a member of an expert community: his/her resources, attitudes, experiences, conversations, and beliefs about mathematics. Perspective is chosen to acknowledge that teacher or researcher is experience and that her/his knowledge is probably broader than that of the students. In this paper, I seek to establish the position that one can articulate mathematical voice more fully by reconstructing one's own mathematical perspective to support rather than to suppress diversity. Thus, the dialectic one engages in is the pursuit and articulation of student voice, and the articulation and revision of perspective in light of student voice, the further development of voice and so on.

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[2] By constructivist practices I refer to the use of small groups, manipulatives, and contextual problems. It is ironic that so many expect the impact of these practices on the content to be only cosmetic, i.e. they believe we can change all of these practices and yet expect to keep intact the mathematics itself.

Although one example can never establish "the case" for oppression of voice, it can illustrate the plausibility of such an argument. To do this, I will discuss my research conducted over the last seven years on a construct I have labeled "splitting" (Confrey, 1994a), and I will focus on one month of a three -year teaching experiment (1993-5) with a group of twenty students who progressed from third to fifth grade (8 to 11 years old). Each year, I spent from three to six months, five days a week, one hour per day, teaching mathematics to these students. The topics involved multiplication, division, ratio, fractions, similarity, scaling, and decimals. In this form of teaching experiment, I designed and taught the curriculum with members of my research group, and conducted frequent assessment. All classes were videotaped. When students worked in small groups, a single small group was selected for videotaping. Post-tests were given each year. During the periods in which I was not working with the children, they received very traditional instruction using mostly worksheets, book materials, and drill and practice activities. The children were from a school which drew on a mixture of backgrounds; some had family connected with the university, both as faculty, staff and international graduate students. Another group was from economically poorer neighborhoods in the town.

In other work, I have articulated the "splitting conjecture" (Confrey, 1994a; Confrey and Smith, 1994, 1995). This conjecture suggests that splitting and counting are two independent primitive roots of number operations (Fischbein, Deri, Nello, & Marino, 1985). Splitting gives rise to multiplication, division and ratio; whereas a counting leads to addition, subtraction and subsequently multiplication as repeated addition and division as repeated subtraction. Splitting has its roots in sharing, mixing, and similarity, etc. The word was chosen because children use it spontaneously and inventively in their early years[3] In addition to the basic splitting structures of doubling and halving, I have demonstrated that the implicit splitting structures, of which there are an infinite number, have a common structure, in which one is the origin, multiplication and division are the key operations, ratio creates the unit, and rate is ratio per unit time.[4]

I have chosen "splitting" for my example, because its roots lie in student voice. In the standard U.S. curricula, the splitting conjecture calls into question fundamental assumptions: 1) it challenges the delay of the introduction of multiplication and division until third grade and the delay of ratio until fifth grade; 2) it creates different mathematical systems for measurement and numeration, which contrast to those built from counting; 3) it connects to geometry in the early grades through ratio and similarity; and

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[3] The disadvantage of this word is that it does not easily translate across languages.

[4] In contrast, in counting structures, zero is the origin, addition and subtraction are the key operations, one is the basic unit, and rate is difference per unit time.

4) it brings rate into the curriculum much earlier. Perhaps splitting's most fundamental contribution is that it portrays fractions as a subset of ratio and proportion.

In this presentation, I will focus on one month of the classroom interactions during which contexts for comparing ratios were investigated. My analysis will focus on the students' strategies for comparing ratios, for interpreting equivalence, and for adding ratios. I will reflect how narrowly these issues are treated within the traditional curriculum. The classroom excerpts will demonstrate both how student voice can be supported within the classroom context through "close listening" (Confrey, 1994c), and the subsequent analysis will show how those voices can encourage critical reflection on the mathematical content itself.

I. Comparing Ratios (Date: February 7,1994)

In order to raise the question of how to compare ratios, I choose the context of voters polls. The reason for this was that the children had opinions, knew about poll's, could gather the data, and would need to compare ratios in order to draw conclusions. Written on the board was a question: "In group A, six children said that Tanya Harding should compete in the Olympics. In group B, five children said that Tanya Harding should compete in the Olympics."[5] The following was the subsequent class interaction:

Teacher/Researcher: Can I tell from that which group , which one had a larger proportion of people? Ah, which one had more people that said Tanya Harding should compete?.

Students: A,

T/R: Okay let me tell you one more fact. In this group there were 9 people altogether and, in this group there were 8 people altogether. So, in which group would you say that more of the people thought that Tanya Harding should compete?

Oliver: They are both the same because they are both three, there they're at least, all, the amount of people, there was always three more.

T/R: Kate's nodding, what do you want to say?

Kate: It seems like . . . (inaudible).

T/R: Both groups have the same. What word should I use? Not same amount, because everyone agrees the first one has more people who say Tanya Harding should compete? What word can we use?

Iris: Percent.

T/R: She says maybe we could use the term percent. Do they have the same percent of people? [a student responds no] Not the same percent.? How are we going to compare those two groups to find out which one seems to have a bigger. . .?

Cameraperson: Would the word "likely" help?

T/R: If I came up to a new person in this group, and they were just like the other people in the group, would this person or this person [pointing to A and B groups] be more likely to say Tanya Harding should compete?

Students: [Little response.]

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[5] In the spring of 1994, Nancy Kerrigan, a skater was attacked and injured as she was leaving the ice during U.S. Olympic trials. Tanya Harding, another skater, was suspected of being involved in the attack. The Olympic Committee had to decide whether to allow Tanya Harding to compete in the Olympics before the case would be resolved in court.

T/R: Not a real clear question. Let me try with some different numbers. Supposing in this group one person said Tanya Harding should compete and I only asked two people, and in this group two people said Tanya Harding should compete and I asked three people, Would you say they are the same now? Oliver?

Oliver: One's one half and the other is a thirds. Two thirds of the people, Two thirds of the people said she should compete, and on A, half so I would say B. Two-thirds is more than one half.

T/R: But Oliver, isn't "two" one bigger than one and "three" one bigger than two, and both have "one" different like you said over here, so . . .

Oliver: So

Jonathan: They are both three off, both three away.

T/R: (laughter) So which of those two do you want to believe in more? So, Ben says they are still the same.

Ben: Not really. Yeah, because, because they are both one under, the number out of, because 2 is one out of two under and 2 out of three is one under.

[Teacher reviews the conflicting opinions. She tries switching the context (using the same numbers) to comparing the case in which for one child, two out of three times a parent approved of the child watching a television show, and a parent of a different child approved one out of two times.]

T/R: Do you agree Ben that to get 2 out of 3 is somehow a bigger proportion of this group than this group [10 out of 2]? Does everyone know 2/3 is bigger than 1/2? We know this one. Now in this one is, if in class B, 5 out of 8, and in class A 6 out of 9, in one of those classes? Is the proportion bigger or is it the same?

Max: The same. I don't understand why it's not the same, because in the other one, it seems, in the TV one like one of the kids just asked one more time. It would seem like it would be the same.

Ben: The only reason there was a difference was because she asked one more time.

T/R: That's a great question. The only reason this one is bigger than this one is the kid asked me one more time. You didn't ask enough Kate. So, suppose Kate asked her parents eight times and told her she could watch show it four times; and Carol asked six times and out of six times her parents told her she could ask and she got to watch it four times. Did either of them get to wath it a greater proportion of the time? We know that they watched the show the same number of times--they both watched it four times--but was either of them more likely to get to watch it?

Students: Yes, Yes, Yes, No, oh yes,

Jonathan: Because 8 is bigger than 6 and she'd have two more times and her parents said yes two more times.

During the next few minutes, three student methods are articulated for comparing ratios. For the first problem, Kai compares 4 out of 6 and 4 out of 8, saying, "Well, you can tell 4 and 8 is one half by looking at it and 4 and 6 is 2/3, and it's harder but 2 goes into 6 three times so that's the third and two goes into four two times so that's two-thirds."

The teacher invites them to return to the original problem. Iris responds, but works with the TV setting while using the original numbers, 6 out of 9, and 5 out of 8. She says: "Well, nine and then eight, times that they ask. Three times they aren't allowed to do it of eight and three times they aren't allowed to do it of nine. And since 8 is one less than 9. . ." Iris said that the bottom one (5 out of 8) was more likely, but on re-examination of the videotape, it is unclear whether her previous answer of "more likely" referred to the likelihood of a child to get a negative answer from a parent since she was using the complements of 5 and 6 in her argument.

Andrew said the top one (6 out of 9) was more likely, and was asked for his explanation. He said, "If another person went to the bottom [the 5 of 8] and they said yes, then it would be 6 out of 9, but if they didn't, it would be 5 out of nine and it would be a lower proportion. I would say the top one would be more likely [6 out of 9]."

What seems evident in these exchanges is that the children were grappling with the problem of comparing ratios. Furthermore, the movement in the class was from a comparison based on the subtractive difference in the numbers to another form of comparison. As the teacher gradually settled on the language of proportion, likelihood, and "More of" vs. "more," the children exhibited four kinds of reasoning related to ratio:

1) their experience with familiar fractions: 1/2 and 2/3.[6]
2) the identification of a common "component"[7] to contrast the relative sizes of the other component (4 of 8 and 6 of 8).
3) a method of additively changing one component into another and then examining how change in the other component affects the ratio. (Andrew changed 5 of 8 into 6 of 9 by adding 1 of 1 which, he argued, made it more affirmative and therefore 6 of 9 was the larger ratio.)
4) a method of examining the complement of a:b (the complement is (b-a):b) to see if they could more easily compare the complements. (Iris contrasted 3 of 8 to 3 of 9 in order to compare the ratios 5 of 8 to 5 of 9).

These are very powerful strategies[8] and with a commitment to student voice and the splitting conjecture in mind, the question for the research team was what problem contexts to explore next. Meanwhile, the students followed up the class discussion with a poll of the students in the school on the same question by grade level. They were encouraged to record their data in a two-by-three contingency table with boys and girls responses in categories along the top and answers of "yes," "no," and "don't know" along the side. Totals were listed along the right hand and bottom edges. The children wrote newspaper stories on the outcomes of their opinion polling.

II. Ratio Units, Episode One: Little Recipes

(Date: February 28, 1995)

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[6] I use the term "fraction" here in that the students' statements seemed to elicit a fractional part. Later in the paper, I discuss in more detail the relationship between fractions and ratios.

[7] In reading the paper, I suggest reading ratios as "a:b" rather than "a/b" unless they are explicitly written that way. I will refer then to a and b not as numerator and denominator but as the two "components" in the ratio. I considered using variable as a description but was concerned that it would carry unintended meanings to the reader.

[8] In most classrooms, even if the student approaches were solicited, they would be ignored as teachers moved to the textbook ways of comparing ratios. In those, typically, a common denominator is found (using only the option of one component) and/or cross multiplication is used and a rule, for a/b to be compared to c/d, if ad. cb then a/b >c/d etc. is produced. In this paper, I will continue to place the typical introduction of these ideas in the footnotes to illustrate that the issue of examining and reconsidering one's own perspective permeates the research enterprise and cannot simply enter as an issue of analysis. Placement in the footnotes also allows the reader the option of staying within student voice without the distraction of traditional perspective.

A decision was made to move to the context of recipes, for it provided an accessible context to talk about varying the components while keeping the ratio constant (Noelting, 1980). Larger initial quantities were used, so that the problem would demand the identification of smaller as well as larger equivalent proportions. Orange and white pingpong balls were used to represent the amounts of concentrate and water in juice mixtures. At first the discussions were about lemonade, but the children later switched to talking about orange juice, because of the color of the balls.

Teacher/Researcher: Imagine I made this much lemonade [a basket is shown with 3 orange balls and 9 white balls] and it had, I wanted to keep it the same. I want to make more of it and keep it the same. What can I do to keep it the same but have more balls in them?

Student: Add some more.

T/R: How are you going to know how to add some moe? What kind of more should I add?

Carrie: Double each part so even though there is more, there is more of each. . . Make the orange balls. . .

T/R: [Teacher realizes there's too few orange balls and adjusts things and then writes on the board three columns labeled lemon, water, and total]

Carrie: To keep it the same but get more, double each number, so its gonna be 3, I mean 6 of the orange and 18 of the whites.

T/R: How many of them altogether?

Students: 24.

T/R: How many oranges?

Students: 6.

T/R: How many whites?

Student: 18.

T/R: (counting balls) Can someone think of another way I can do it?

Andrew: Just add three whites on every time you add an orange.

T/R: Seth says how that would work. Why do you think that works Seth?

Seth: I don't know, it just seems to work.

Andrew: Just add three whites and one orange, because that's sort of the pace that the orange is one-fourth of the total, and so every four you would take one orange ball and three whites.

Iris: 3 is one fourth of twelve and. . . .

T/R: So here we are back here again. Let's stop. I have too much lemonade and I want less lemonade but I still want it to taste the same. What can I do? What would you do.?

Amy: Let the lemons sit there and take it [water] out.

T/R: You are going to take some of them out. How shall we take them out? (pause) Should we take out some of the water? OK, supposing I took out that much water (2 balls)?

Student: But it would be more sour.

T/R: So I can't just take out the whites, so what do I do?

Student: Take out one lemon.

T/R: Does that end up tasting the same?

Students: No.

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T/R: That would be less sour, more watery. Okay, how about if I take out one of each, is that going to taste the same?

Students in chorus: NO!

Skye: Just take out three whites and one lemon.

T/R: And that's going to be the same? How many balls in there? Let's count. So he says that I will have eight whites balls. How many of each? Six . . . and two and .. . .Pierre do you disagree? I want it to be the same. I really want it that way. Max I need less. I can only drink one glass. It's way too cold outside to drink more than one glass. What can I do now.

Max: I'm not sure.

T/R: Think of it, I have a mixture. Tell me something I could try.

Max: One lemon and three balls of water.

T/R: Three frozen water balls. Why did you decide that?

Max: Because, it just seems like the thing to do.

Ryan: Because you need one lemon and three waters to keep it the same.

T/R [completes the chart with the students calling out pairs.]

T/R: (we switch to orange juice and review what we did) All of you have to drink my lemonade. I want to make more than 18. What can I do? I want three different ways to make more than 18. I see three hands. Raise your hand. Let's see.

Iris: Double 18 same way.

T/R: 36, 12, altogether, 48.

Kate: Add three to 18.

T/R: Which is?

Kate: 21.

T/R: And this one?

Kate: Add one to get 7.

Ryan: Triple the 18 and the 6.

[They work out the arithmetic]

T/R: We tripled it, we doubled it, we added three and added one to each but I don't know how to code this one any other way.

Max: Add six and then add two. I think. (works out the arithemtic)

Kate: Add nine and then add three.

Ryan: You gave us an example and we changed it into so many things.

In this exchange,[9] the students proposed a variety of ways of discussing how to keep the taste the same. All the children seemed comfortable with the idea that they can double or triple the recipe to maintain its taste. This was a simple confirmation of the intuitive role of splitting for the children. One child was able to find a way to increase the recipe by adding three to one component while adding one to the other. He referred to this as "pace." Other children seemed to grasp this idea and extend it to include subtraction of components, although it took time and varied across students. Later it became clear that finding the "pace," as Andrew called it, proves to be a challenge for many, but its importance was apparent to all, and its use in adding and subtracting to make more or less while keeping the taste the same was widely accepted. Notice that in this exchange, the table became established as an important record-keeping device.

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[9] In a typical class, equivalence is discussed extensively in the context of fractions. In this context, most teachers show equivalent fractions by demonstrating with circle diagrams or with fraction bars equivalence in area or in length. Quite quickly then the child is taught to reduce fractions by finding common factors and the question of what the factors represent is seldom examined in detail. If it is, it might be liken to an act of cutting all the equal sized pieces into the same number of parts which is described as multiplying the numerator and denominator both by the same number, n, but since n/n equals 1, it is viewed as multiplying by an identity which doesn't change the number.

In examining Andrew's proposal, it became clear to me that with my previous work with functions, I could describe the approach as covariation (Rizzuti, 1991; Confrey and Smith, 1995) and that they were working with a precursor to a linear rate concept. Schorn (1989) described this as "so much of this for so much of that." In other work, I have argued that this (3 to 1, in this example) should only be called rate if the child can imagine it increasing (to 4 to 1, for example) or decreasing (2 to 1, for example). In a setting in which only a single value for the ratio is possible (as pi is the ratio of the circumference to the diameter) a description of ratio as "the invariance across a set of proportions" should be applied (Confrey, 1994a).[10]

The children were asked to work in groups with manipulatives to see if they could adjust these recipes to produce charts with 12 entries all with the same taste. A new description emerged: "a little recipe." In one group, the girls had made sets of unifix cubes in the proportion of four to one. When asked how they knew the juice would taste "the same," they said they could make it by putting together sets of "little recipes." This term became a favorite of the class and was used throughout the instructional program to refer to the smallest whole number equivalence for a given proportion. A second description that emerged was of "basic combination". These descriptions as recipes and combinations communicated the idea of a set of ingredients or components that belonged together and the adjectives "little" and "basic" connoted their use as units.

I would first propose that the "little recipe" be consider to be a unit, defined as the invariance in a repeated action between a predecessor and a successor (Confrey, 1994a). Secondly, I propose it be named a "ratio unit" for it describes both the underlying ratio (as a member of the set) and acts as a unit for increasing and decreasing the combinations. An interesting question is whether this is an additive or a multiplicative unit. I will suggest that it functions as both, and that this feature is its defining characteristic. The ratio unit can also be described as a "basis vector" in linear algebra for it allows one to span a vector.

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[10] I recognize that this is a reversal of modern textbook definitions which tend to say that 1/2, 2/3 and 3/6 are different ratios al "in the same proportion". I agree with Fowler (1987) who argues that this is an "arithmetized" view of ratio in contrast to the Greel perspective, and, as such, loses the essential characteristic of invariance. It presumes that the fraction line notation (/) conveys the operational relationship of a to b, and treats numbers as abstracted entities which can be compared without any knowledge of the means of their construction. I would also point out that the terms ratio and proportion are hopelessly conflated in English usage. We talk about keeping the same ratio across proportions, yet we also say two ratios are in the same proportion. In the context of a recipe, the question is for a table of entries, (1,2); (2,4);(3,6) etc. is this a set of proportions with one ratio or is it a set of ratios with one proportion? I prefer to use the term "ratio" to refer to the invariance across a set of proportions, and hence there is one ratio (a, b) that underlies that set of proportions. However, I am less comfortable saying that (1,2) and (2,4) are different proportions, since many will not understand such usage. Perhaps, as a compromise, I could use the terminology that (1,2) and (2,4) have the same rat but are different combinations. In contrast, (1,2) and (1,3) are different ratios and different combinations.

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Others have pointed towards this same construct in ratio and proportion (Lamon, 1994; Steffland, 1991; Steffe, 1994; Kaput & West, 1994; Lesh, Post, & Behr, 1988) and have focused primarily on its use as an additive unit, in building up to larger combinations. However, due to the tendency to focus on the task of building larger combinations and on only viewing multiplication as a form of repeated addition, most researchers have overlooked the process of finding the "little recipe" within larger combinations. They seem to assume that doing so would involve a process that is simply the reverse of "building up". In contrast, I will describe how the identification and development of the ratio unit gives it a "split personality" (additive and divisional) and, as such, it serves as a critical bridge between counting and splitting worlds. How the ratio unit emerges from the childrens' discussion is a springboard to understanding how the ratio unit is a multiplicative unit.

III. Ratio Units, Episode Two: Finding Ratio Units

Finding the little recipe[11] in the context of being presented with larger quantities posed a serious challenge. One group of girls that same day was working on reducing the amounts of original mixtures containing 15 water balls and 20 lemon balls. Seeing 5 as a common factor, they tried to subtract five from each as the way to reduce the recipe, but rejected it quickly. Then they considered splitting the components into two halves but realized that 15 wasn't even. They considered splitting the balls into thirds, and started dealing them out into three piles. They had split the water balls evenly into three piles, but they realized that the 20 lemons didn't divide up evenly into threes. This is an excerpt of their discussion:

> What goes into both? five. Split them into five piles. I'll deal out the water and you deal out the water. It sounds strange dealing out water. Three, three, three, this has four and this has three [they realized that two lemon piles were missing a lemon] They look down and find it on the floor.] Oh, good, we did drop two. . .. When we take away three waters, that's 12 and four of these, so that would be 16 . . .[At this point they realize that the little recipe is 3;4.]

The children initially saw the importance of the number five, but their first conjecture was that it would show up as a value in the little recipe. Reconsidering, they switched to splitting, and after trying to use two- and three-splits unsuccessfully, they made piles of five. I believe that for the children, doubling and halving, which I would call two-splits seem to be a very legitimate action which maintains the ratio. The five-split doesn't actually feel like a split to them here, rather, it is thought of as making equal piles.

It appears that two interpretations of this are worthwhile. First, when the children can find a way to split the recipe, it is straightforward for them to assert the equivalence of the final and the original recipe. No visible record of the value of the split is retained. Second, when a split is not an obvious choice for them, they resort to creating piles, in which the number of piles is equivalent to the split. Although these two actions, and dividing by a common factor, are all coded mathematically the same, the differences among them are significant to the children.

How should one interpret the children's claim of equivalence between the final and the original recipe? This equivalence, established most firmly through splitting, is the essence of the meaning of ratio. It can be interpreted variously as recursion, similarity, or stretching, and its expression creates a meaning for division (partitioning) and ratio. In interpreting the next excerpt, I will explain how a meaning for multiplication that is in contrast to repeated addition is also created through the students' voices.

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[11] In the traditional curriculum, finding the little recipe means simplifying fractions. It is usually treated 1) as an issue of tidiness or completion of a process, and 2) it is undertaken by factoring and "canceling" or "finding names for one."

(Date: March 1, 1994)

For the next two days, the students worked extensively with the recipe context and the problem at hand was how to find the ratio unit.[12]

In this class, we began by reviewing the problem from the day before. The students restated the problem for students who were absent and one girl stated the problem as "we were making orange juice and those were the ingredients and we were trying to get the exact taste we wanted and we worked a real long time trying to get the exact taste and so now we want to keep it and make a larger quantity of it" (Claire). The teacher and class reviewed that if we added more water balls it would be more watery and if we added more orange it would be more orangey, but we wanted to keep it the same. Previously absent students were asked how the recipe could be increased while keeping the taste the same.

Naomi: You could add the um-m, a third, um-m add some orange juice, and add some water balls but the water balls would be a third of what the orange juice would be

Teacher/Researcher: Beautiful, nicely said. Jon, does that make sense?

Jonathan: It is kind of a third, because six is one third of eighteen.

(Discussion with a student who doesn't respond and we discuss adding one of each. They decide they can't)

T/R: Naomi you said it was about a third, the water balls to the oranges, that the water balls are one third of the orange juice. So, if I add one water ball, how many orange do I need to add?

Jonathan: I think you add a three.

T/R: Three oranges, so if I add one water ball I'd add 3 oranges. Does that make sense to everybody? Naomi just saw that and said it looks like a third. The hardest problem you all were having the other day was finding out what that initial amount was. How did you find out this was the same as one-third? What were some ways of finding this out? How did you know this was the same as one water ball to three oranges?

Carrie: If you think of it as just a number, 6 is one third of eighteen

T/R: 6 is one third of eighteen because 6 fits into 18 three times

Kate: We took blocks and divided them up into equal piles.

T/R: OK, so we will try that one in just a second, okay? How would taking blocks and dividing them into equal piles look here? (points at board)

Arna: We did the same thing, and we have, one was water and one is orange juice and found out it divided equally into little recipes and ended up being three orange to one water, and just (unintelligible).

T/R: You divided equally into little recipes and can you tell me what that means to you?

Arna: And then we wanted to make more lemonade so it was in the same proportion.

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[12] As instructor, I was expecting that finding the ratio unit was just simplifying fractions--prime factoring and factoring out a common factor. Slowly, I came to the realization that it was not so straight-forward to the students, or to me once I began listening to them. This challenge is typically neglected because of the tendency to always start small and simple and thus emphasize building up rather than splitting.

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T/R: How did you find those little recipes?

Arna: We put eighteen of [orange] and six of water.

T/R: [Draws it with dots in two rows vertically over each other]

Arna: We tried to divide it into groups so there were the same amount of orange balls and water balls.

T/R: And then they said, listen to what their problem was. How can we get this into groups that have the same number of white balls, no, how do I say it?

Arna: the same number, no each group has to have the same number of water balls and orange balls.

T/R: Does that mean it has the same number of water balls as thre are orange balls?

Arna: No, we just tried four to one but it didn't work so we tried three to one. . ..

T/R: [draws it and it, the class applauds and says the drawing looks like dogs with three eyes] They said each one of these is a little recipe. Once you knew what the little recipe was then what did you do?

Arna: We could combine the little recipes to make bigger ones.

[During the next few minutes the children discussed ways of building up and down given the smallest recipe. Also, there was a strategy of halving and one of taking one-third of another number)

R/T: Could you get directly to 1 and 3 by looking at the numbers. Is there a way to get directly to one and three

Oliver: Divide the 18 by 6 and divide the six by one, no by five, by . . .

T/R: What do you divide six by to get one?

Oliver: 6.

T/R: Does it seem like you should divide it both by the same thing? [Oliver divided 18 by 6 and 6 by 6.]

In the next problem, the children were given 14 balls of water to 21 balls of orange. They were asked to find equivalent proportions.

Claire: You could make it bigger, you make 14, 28, and over there 42.

Teacher/Researcher: So, tell me what she did

Trev: She added or she doubled.

T/R: That's always a good method

[garbled exchange]

Carrie: 14 is 2/3

T/R: 14 is 2/3 of what?

Carrie: 21. So add two to the fourteen and add three to the fourteen. The last time we had one third and we added one and three so this time it would be 2 and 3. But I don't know which to add to which.

To find the little recipe, the majority drew dot diagrams or worked with manipulatives. Their goal was to make identical groups, or as one child put it "to divide it into groups so there were the same amount of orange balls and water balls". This description conveys a significant part of finding the littlest recipe that is often overlooked. This is the splitting or partitive act that must produce: 1) the number of partitions that can be applied to both sets without a remainder; and 2) identical ratio. This activity is not algorithmic, just as its more formal method which depends on prime factoring is not algorithmic.

In addition to my earlier claims that previous work on the ratio unit has neglected the creation of that unit, I would stress again that in the creation of that unit, we co-create partitive division and a new meaning for equivalence. Equivalence in this setting is not based on equal length of the original combination and the ratio unit. In the dot drawings for 6 and 18, the little recipe is equivalent to the original combination because all the little recipes are identical, not because there are six of them. On a graph, we could say the equivalence occurs because when the vector (6, 18) hits on a lattice point, so does (1, 3) and it is the smallest unit that does also. The little recipe is a unit that measures the larger combination, and I would reiterate, that as a result, there are direct ties to recursion and similarity.

A second approach to find a:b was to describe na:nb as follows: na is a/b of nb (such as 6 is 1/3 of 18 or 14 is 2/3 of 21). This description needs close listening and careful interpretation, because mathematically modernly trained ears hear it differently than is intended by the children.[13] On reflection, I would interpret their descriptions as follows: One third of eighteen means that eighteen is split into three parts. Expressing it as a split, children will say, 18 split among three is six. (18/3= 6). Another form of this same equivalence might be the claim: 1/3 of 18 is 6. I want to suggest that this is a primitive expression of the multiplication that goes with partitive division--multiplication as "1/3 of." Thus, we write 1/3 of 18 is 6. That is, the meaning of "1/3 of" to these children is not as a fraction but as "1/3 of" as an operator, a splitter. "1/3 of" means dividing into 3 equal piles (Dienes, 1967).[14]

Ultimately, I believe that both of these expressions can be even better coded using a ratio box[15] which is written as

Orange balls	Water balls
18	6
3	1

To later make the claim that 14 is 2/3 of 21, they are splitting 21 into three equal groups (or taking one third of them) and taking two of them or doubling the amount. Ratios, as opposed to fractions, are operators on a set of numbers. This description allows a ratio unit to be found[16,17].

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[13] I believe my training led me to hear this as an application of the algorithm for multiplying fractions, and thus to believe that children were importing this algorithm from home instruction. Thus, 14 is 2/3 of 21 means 2/3 • 21 = 14. I rejected this interpretation after reviewing all its occurrences on tape because the students who offered it were so numerous and varied in their mathematics preparation. Furthermore, most people were taught that "of" means "multiply" without ever examining the reasons for that translation. In contrast, I now believe that its spontaneous occurrence signals a different construct of multiplication which eventually leads to the algorithm for rational number multiplication.

[14] For years, I have critiqued the claim that there are two primitives for division but only one for multiplication (repeated addition). I now understand that the difficulty we had finding the second primitive for multiplication came from trying to cast multiplication as the inverse of division, in the form of: if c/b = a then a x b = c. Instead I am arguing that partitive multiplication is found by examining partitive division in the form of c/b = a, and casting the inverse operation in the form of 1/b x c= a.

[15] Building on the work of Vergnaud's isomorphism of measures vs. functional relation analysis (Vergnaud, 1983, 1988) of multiplicative structures, Peled and Nesher's (1988) further application and Mechmandarov (1987), I used a 2x2 matrix to explore ratios with students.

[16] In conjecturing these new interpretations of how multiplication, division, and ratio are related, I realized that they needed to be written in the form of arithmetic equations which bring out the role of the equals sign (=). Thus operations are not just codes of actions, but are codes of actions within equivalence relations.

[17] In traditional instruction, ratios are notated the same as fractions. They are introduced later, in isolation, and without any forms of operation except comparison, those all have been "covered" in fractions. Thus, any inconsistencies in the two systems are kept in abeyance and when students try to integrate these and fail, the failures are labeled misconceptions.

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The advantage of this approach over the dot drawings is that eventually it leads to a way to find a combination for any value of a component, and one is not limited to search for appropriate lattice points. For example, it will assist them in making the claim that one-third of four is one and one-third.

The third strategy used by the children was to divide the same number into both of the components to get the ratio unit. This strategy was used routinely by only two students and did not really catch on among the others despite the fact that it produces the littlest recipe most readily. Difficulties were seen repeatedly as children recognized a value of identifying a common factor, p, between the components but then tried to use that factor as a component of the ratio unit making the ratio unit either 1:p or p:p (just as the two girls had tried to subtract fives for 15:20). However, this never happened when they used the language of splitting. Thus I suggest that the word "division" for many of these students still referred to quotative division (repeated subtraction) only and thus it was of limited applicability in finding the little recipe.

IV: Introducing Graphs of Ratios

Students were introduced to graphing of ratios in two dimensions as vectors of the form (a, b). My decision to do this was based primarily on the fact that it allowed for a visualization of ratio as an invariance across a set of proportions (or combinations). Other considerations included: 1) seeing the ratio construct as strongly tied to similarity and wanting to build those bridges in the elementary grades; 2) wanting to support the comparison of ratios in which both components of the ratios could be used equally as a means of comparison (as we had seen students spontaneously do in the context of polling); 3) seeing this as a rich extension of the use of tables and a precursor to algebraic functions and slope; 4) wanting to explore how combining samples could alter the ratio relation; and 5) seeing the comparison of ratios as a rotation in the first quadrant, an anticipation of trigonometry.

The introduction to graphing was done using the context of recipes. The axes were labeled number of water balls (horizontal) and the number of orange balls(vertical).[18] The students are instructed in how to graph points that "stand for certain combinations," and the origin was identified as meaning no water balls, and no orange juice. The term, ordered pair, is introduced as (a, b) using the example of 6 water balls and 8 orange balls; "a" is defined as how far to go out and "b" as how far to go up." The children are asked to form groups and plot the points for six of different recipe problems (multiple examples per mixture) using different colors.

(Date: March 7)

In order to help students to see how to use graphs to compare ratios, they were given another polling problem. It had been a particularly snowy winter, and they were asked whether they were hoping for another snowstorm. The class's response was 3 "yes's" and 12 "no's."

The teacher/researcher asked, "Supposing I have a group with 7 people who say yes and 21 who say no. Listen to the way we can say it now: "In which of these classes does a greater proportion of the people say they want a snowstorm. I want to know if in [our] class or in this class [points to two different sets of data], which has a greater proportion of the people who say they want another snowstorm. In which class are you more likely to get someone to say 'yes, I want another snow storm'." One of the children quickly identified the little recipes and said that one yes out of four is less affirmative than one yes out of three. The teacher/researcher asked them, "how would this look on a graph?" The children generated the points (1, 3). (2, 6), (3, 9),..., (6, 18), drew in a blue line, and compared them to the points (1, 4), (2, 8), and (3, 12), whose line was drawn in red.

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[18] In other contexts (polling), these axes could be labeled using parts and totals.

Teacher/Researcher: What do you notice?

Kat: There are more steps and it's not as steep.

T/R: If you were walking which would you rather walk up? The blue one? It's not as steep. How do you see that the blue class was more likely to say yes? How do you see that the people are more likely to say yes?

Trev: Because that one slants out more towards the yes.

T/R: It comes out that way. What is the meaning of the point (10, 3)? That's really "yessy." At what point is a combination more "yessy" than "noey?" How can I find the place where you can say it's more yessy or more noey? Lets try it with points--How about right here? [points to (1, 5)]

T/R: (1, 5).

Students: More noey

T/R: (3, 2)

Students: More yessy.

T/R: Where does it change?

Skye: (2, 1). More yessy. I mean (2, 1).

Max: It changes at (1, 1), or (2, 2), or (3, 3), (4, 4), (5, 5), (6, 6).

T/R: Do you see how you can use this picture to compare. . .?

Iris: When they both meet on the same line...

T/R: If you look at 12 where they both meet at the point in time, we know we have the same number of...?[points to (3, 12) and (4, 12) where there are 3 yes's and 12 no's, and 4 yes's and 12 no's, and the points are in horizontal alignment]

Iris: No's.

T/R: The same number of no's.

T/R: What you are doing here is really important. Do you see how much information you got out of the graph? Out of those pieces of data we made all kinds of conclusions... Iris, you could ask if there is a point of time when they are this way together [indicates a vertical alignment of points].

Iris: Compare [at] 4 yes's and [you get] 12 and 16 no's.

In this segment, the children explored the graphs as a means of comparison. One student seems to see the comparison rotationally, oriented more towards the yes. The teacher/researcher switches from a comparison of two graphs of different ratios to the question of when any single combination becomes more weighted towards yes or no. Students also recalled and visually applied the strategy of comparing ratios by seeking out when two components were the same, and comparing the other component of each combination.

V: From Voice to Perspective

The issue raised in this paper is "What are the implications of the students' activities and approaches, the students voices, for our perspectives on mathematics." The analysis presented thus far entailed a certain amount of interpretation of student voice and the development of such constructs as ratio units, splitting, and partitive multiplication. On the surface, the data suggest that the approach is provocative and potentially fruitful.[19] But provocative and potentially fruitful for what purpose? To teach the rational numbers? But are the rational numbers well-defined and fully understood by most of us? Is there an agreed-upon endpoint toward which to be aiming? Asking such questions critically is, for me, an essential part of examining our perspective. Asking these questions in light of acknowledging the inventiveness of student approach is what is meant by the voice-perspective dialectic. If the analysis stops here, then we fail to acknowledge the tensions, choices, ambiguities, and interpretations which are part of the mathematical enterprise. The analysis should not stop here. To me, stopping here and bringing children into alignment with adult perspective is a major way that mathematics becomes oppressive rather than expressive to children.

It seems clear that the data presented above support the view that children can operate intelligently with ratios, especially if they are provided access to appropriate representations (data tables, ratio boxes and two dimensional plane) within interesting and familiar contexts. This research challenges the assumptions of many who claim that proportional reasoning occurs developmentally later than 10 years old. It seems to me that our challenge is to make judgments about what ideas are important for children and to then follow their approaches to create a developmentally sound curriculum that also leads to profound and accessible mathematical ideas. Thus, the analysis of the rational numbers I seek to undertake in the next section must 1) allow us to explain and support sound student constructions; and 2) anticipate significant achievements in mathematics that connect to those constructions. In doing this, existing curricula should be artifacts to be considered but they should not hold particular sway over the decisions.

In the foregoing approach to ratios, ratios have been treated as combinations that lie along the same vector. Moving to the two-dimensional plane to express ratio exhibited one clear advantage explicitly acknowledged in the splitting conjecture, namely that connections to geometry are made accessible earlier. The importance of this goes beyond the exploration of the properties of geometric figures (itself a topic of considerable value to children) to include the analysis of figures in relation to their symbolic algebraic presentation (anticipating analytic geometry; see Dennis, Smith, & Confrey (1992) for a discussion of the role of similarity and proportional reasoning in the development of functions.)

A question that comes with this claim is what are the appropriate mental operations that would benefit from being formalized on this two-dimensional plane.[20] These should be the set of all operations that 1) assist one in working competently with the desired contexts; and 2) connect to other operations from previous and future mathematical pursuits. One approach would be to seek possible metrics or structures[21] that contribute to this analysis and interpret the operations from within them. The children's operations needing to be explained are: 1) comparing ratios; 2) finding the ratio units; 3) relating ratio units to their equivalent combinations; and 4) refining the sets of possible combinations. In doing this analysis, I identified three possible metrics (see Figure 1). Each sheds light on the children's actions, and most allowed me to convert what most researchers label misconceptions of children into potentially productive strategies. The standard related properties and operations one seeks to connect to are: 1) multiplication and division, 2) addition and subtraction, 3) distributive properties, and 4) the standard ordering properties--less than, equal to, and greater than.

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[19] In the post test, the 9-10 year olds exhibited far less additive strategies than reported by 13-15 year olds of CSMS. 15% of our sample used additive strategies on Mr. Tall/Mr. Short compared to 47% in CSMS. 30% used additive strategies on the L's problem compared to 42% for CSMS (Hart, 1988).

[20] The question of what level of formalization is desirable is debatable, and needs to be subjected to a process of social/political discussion that is more extensive than I can undertake in this paper. However, I do seek to demonstrate some of the available options and to indicate how they shed light both on the student methods and on our own tacit assumptions about elementary mathematics.

1. By creating a vertical fracture[22] of the plane at x= 1, one creates a one-dimensional line which each of the ratio vectors (a, b) intersects at a distance b/a from the line y=0. The reduction of the two-dimensionality of the plane to a one-dimensional line creates a metric that behaves like the fractional number line, on which, for example, the vector ratio (3, 1) intersects at (1, 1/3); the vector (3, 2) at (1, 2/3) etc. If the vectors are viewed algebraically as y=3x then when x=1, y=1/3. In order to view this as fractions (as) parts of wholes, the [total] needs to be represented on the x -axis and the [parts} on the y-axis. Equivalence on this metric means being the same length. Thus, one definition of a fraction is "the application of the ratio vector (a, b) to the quantity of unit one." Thus, when we see b/a as a fraction, it could be written as (b/a) of (1). Thus, when one sees the fraction 3/4 and the fraction 2/3, by convention, one can assume that they have a common unit of one. As a result, addition of fractions with unlike denominators requires finding a common denominator to protect the meaning of equivalence as equal length.

However, as we can understand from the students work with ratios in the two-dimensional plane, working with fractions does not have to be confined to the line x=1. Using this approach, one could view the comparison of 3/4 of 5 and 2/3 of 6 as a comparison of one fracture at x=5 with the vector (4, 3) and the other at x=6 with the vector (3, 2). What makes these fractional approaches is that the "wholes," of which the fractions are the parts, are specified by the values of the fracture lines, and the parts acts as ratios via the ratio vectors. Thus the distance from y=0 to the intersection of each vector with its corresponding fracture line forms the basis for comparison.

2. A metric on the first quadrant of the plane can be constructed by drawing a quarter circle with the origin at its center. The ratios can be ordered according to the intersection between their ratio vectors and the arc, making rotation the basis of the ordering. This ordering of the ratios will correspond to the ordering on the rational number line, however the dispersement of the ratios will differ. In the work with the fourth graders, drawing their attention to rotation as a means to compare ratios anticipates these

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[21] By metric, I am referring to a way to formalize the comparisons between ratios. Since I am not assuming an external real line, this is not intended to be a formal axiomization of this idea, but its intended to allow us to examine the impact of the genesis and choice of metrics on the definition of operations.

[22] A horizontal fracture will also create the rational line.

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developments. Later in the semester, the children designed and built wheelchair ramps, an activity in which the angular measure and corresponding ratios were explored in some detail. Connections to trigonometry can be later developed, via the vertical projections from the intersection of the vector and the arc, which is of course $\sin(\theta) \times$ radius.[23]

3. Finally, one can examine any single vector used as a means of generating equal ratios across different combinations for its potential metrics. It is possible that there are two competing metrics on this vector, each with a different meaning for equivalence. The first metric comes out of adding ratio units to produce larger combinations. Describing this form of addition can be accomplished using vector notation. It can be written that the students are adding $(2, 3) + (2, 3) + (2, 3)$ to get $(6, 9)$ and vector notation suffices. However, after the ratio units are combined, the student wants to make the claim that $(6, 9) = (2, 3)$, which is not true in vector notation. Vector notation requires students to write this as $(6, 9) = 3 \times (2, 3)$, introducing the idea of a scalar, a "dimensionless number." In contrast, in ratios we wish to allow the student to write that $6:9 = 2:3$, thus legitimating a different kind of equivalence. That leaves us needing to explain what happens to the scalar 3 in the vector notation of this equivalence relationship. I claim that the 3 is the split.

Splitting, or division into parts, creates a set of identical groups, and this congruence is itself the basis for the claim for the equivalence of the ratio unit and the larger combination. I have further posited that this equivalence creates the meaning for the operation of partitive division and subsequently for the idea of a partitive multiplication of the form 1/3 of 18 is 6. These operations of splitting, partitive division, and partitive multiplication form the operations that will allow us to describe the second metric on the vector, however, doing so in depth is beyond the scope of this paper.

Thus, I can now clarify my concerns with the definition of ratio by Thompson (1994, p. 190), in which ratio is defined as a multiplicative comparison of quantities. It is my contention that the equivalence of combinations that creates the invariance we label "ratio" comes from splitting. Furthermore, partitive multiplication develops as the inverse of splitting, and thus this multiplication arises out of ratio, not the other way around. As a result, Thompson's definition describes a formalism only, and his relative placement of ratio and multiplication is phylogenetically and ontogenetically out of order.

My claim is that in each of the systems, we see variations in the meaning of equivalence, multiplication, division, addition, and subtraction as voiced by the children. These differences indicate that there are options for defining the meaning of operations. Depending on which option one selects, one enters into a variety of mathematical subfields: vector arithmetic, trigonometry, linear algebra, fractions, etc. However, in all of them, the concept of ratio is essential. Preparing students for operating formally in only one of these, (namely, fractions), and disallowing the others, 1) sends an inappropriate message to students about the character of mathematics, 2) gives them only a narrow preparation, 3) discourages diversity in approach and thought, and, later 4) makes the reintroduction of these other approaches unduly formal and awkward. Furthermore, as demonstrated through the examples given here, if one seeks to develop a contextualized mathematics, one which does not sever its ties to experience and activity, there is a need for moving among these different structures flexibly and knowledgeably.

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[23] There is also an important question to be raised about vector addition. One can argue that when combining samples, the children should use the notation $(a, b) + (c, d) = (a+c, b+d)$ rather than $(a/b) + (c/d) = a+c/b+d$, but that solution becomes less satisfactory when one witnesses it being used as a way to compare ratios as in the first excerpt where Andrew adds $(1, 1) + (5, 8)$ to get $(6, 9)$ and concludes $(5, 8)$ is a lower ratio. It appears some form of a metric ordering is emerging here.

The point here is not to resolve all these issues, but to point to the need for mathematics educators to examine more critically their own beliefs and reasons in light of student voice. Our current tendency is to try to change the children rather than to change the mathematical notations to express the needed distinctions.[24] In mathematics education, we all too quickly label a genuine intellectual dispute as a child's misconception. Although the mathematics education, theories about epistemological obstacles and theorems-in-action invite us to engage in such epistemological investigations, seldom has their pursuit ended up challenging the conventional mathematics. One must wonder whether this is a product of the perspective of the observer rather than the voice of the student. If we want critical thinkers, we need to acknowledge that doing mathematics is creating a composition of reason and algorithm (procedure) and that doing so combines convention and craft (Shapin & Schaffer, 1985).

VI: Discussion and Conclusions

The value of the splitting approach needs further assessment, clarification and refinement. To date, it has been tried in a highly experimental setting to allow the development of the conjecture and its related implications. Basically, the teaching experiment has demonstrated the plausibility that splitting can be developed in parallel to counting and that when ratio and proportion are used as the advance organizers for relational thinking, there is: 1) considerably less evidence of the inappropriate application of additive strategies for ratios, 2) an compelling transition to algebra through rate of change and functions (Confrey & Smith, 1995), and 3) strong ties to geometry, in particular to similarity and trigonometry. Furthermore, this approach has certain advantages in that: 1) builds from children's knowledge; 2) relates that knowledge to relevant cultural experiences; 3) involves the use of fundamental tools; and 4) invites students who may not have efficiently mastered addition and subtraction back into the curricular stream.

In this paper, I have further argued that we, in mathematics education, have accepted a model for teaching the rational numbers that gives the arithmetic of fractions, as modeled on the one dimension number line, precedence over the varied arithmetic and geometric interpretations of ratios. In doing this, we have inadvertently narrowed the intellectual preparation of our students, discounting student strategies that would support explorations into mathematical subfields such as trigonometry, vector operations, and linear algebra. Later, we try to reopen these territories for children. When we do, we encounter resistance, because the children have internalized our views that there is only one correct way, and no longer believe they have a claim on mathematical thought. Furthermore, our introduction to these ideas

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[24] One of the reasons research on children's understanding and use of programming is so compelling is that given that programming applies directly to the design of new technologies, one feels free to change the programming language to produce desirable outcomes. In this sense, programming is a living language. Mathematics is considered by most an ancient language, to be interpreted or translated but not changed. This seems unfortunate for it inhibits its possibilities of being an expressive language.

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at the higher grades tend to be even more distant from everyday thinking and common contextual problems, and relies almost exclusively on the manipulation of symbolic languages. As we privilege a formal presentation over the use of applications and physical tools, we concurrently allege that only those who survive the gyrations of this system, merit passage through the mathematical gateway.

In this paper, I have made the claim that these practices are oppressive. I now can express this position more clearly by specifying an epistemological analysis of my use of that term, and claiming that this oppressiveness in fact has a long socio-cultural history.[25] In other work, David Dennis and I have documented, in the history of mathematics, the suppression of the use of multiple representations (Dennis & Confrey, 1995) and physical apparatus and tools (Arnol'd 1990; Confrey, 1993). In a recent paper, Otte (1993) has analyzed the time period of Fourier and claimed that during this period, pure mathematics was established as a specialization which parted ways with both physics (which assimilated non-observable but empirical constructs such as electricity) and technology (which became engineering). His analysis stressed the way that the dominant forces on mathematical development prior to this period--mechanics and geometry--were replaced by an emphasis on axiomatization and arithmetization.

If this analysis stands, then my analysis of rational numbers stands as an example of exactly these values from pure math. The elementary curriculum can be viewed as implicitly dominated by arithmetization and the reduction of the ratio space to the single dimensionality of the rational number line. Furthermore, a press for axiomatization makes the decision to neglect the multiple meanings of operations *an intentional choice*: a suppression of diversity in return for early exposure and training in formalization.

Abstraction within this value structure loses its power as an expression of the similar in apparently dissimilar problem situations, because formalization into symbolism that denies explicit connections to context becomes preferred. In Vygotskian terms, one could say that the semiotic portrayal of mathematics becomes its defining characteristic and its value as a tool *for action* is diminished. Abstraction, which should be the pulling together of context and activity with symbolism and language, develops more as a pseudoconcept.

My claim is that within an epistemological environment that emphasizes formalization, axiomatization, and arithmetization to the exclusion of context, multiple representation, and tool use, the expression of student voice is suppressed. I have taken the position that the suppression of legitimate diversity in student approach to mathematics is an act of oppression and permits mathematics to act as a filter. By only acknowledging a narrow range of student strategies and then portraying only these approaches as "correct" or "true" rather than as choices or alternatives, children are discouraged and alienated from mathematical thought. Only those who find the "playing" of the game according to an external authority's rules attractive, or have access to individuals who demonstrate a fuller view of mathematics, persist in such a system.

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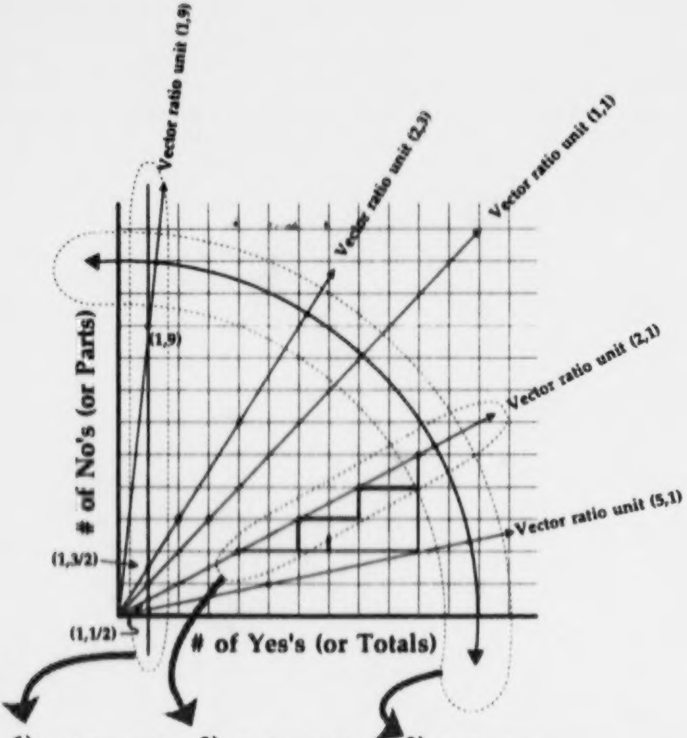
[25] I would cast this analysis as feminist epistemology because of its use of the voice of oppressed groups to establish alternative perspective in mathematics. In addition, reexaming history in light of such concerns about oppression has a strong precedent in feminist writings.

In closing, I wish to suggest that mathematics educators have a critical role to play in creating a different context for the learning and pursuit of mathematics. However, to play this role, we must be willing to engage in deep examination of own our beliefs, allegiances, purposes, and assumptions about mathematics as much as about its learning and teaching. We must draw upon our rich heritage in mathematics while viewing our commitments to educate as taking precedent over our training as mathematicians. We must collaborate with a variety of mathematicans, those in pure mathematics and also those in applied mathematics, and those who use mathematics in their work, both in and out of the academy. We must raise hard questions about what to teach and how to teach it, and in doing so, our first commitment must be to the well-being and future growth of the children.



Vector ratio unit (1,9)
Vector ratio unit (2,3)
Vector ratio unit (1,1)
Vector ratio unit (2,1)
Vector ratio unit (5,1)
(1,9)
(1,3/2)
(1,1/2)
# of No's (or Parts)
# of Yes's (or Totals)

1) x=1 producing the rational number line

2) Vectors (a,b) creates metrics among equivalent combinations

3) A circular arc creates an ordered set of ratios by intersecting with the ratio vectors

Figure 1

Three Metrics on the Two-Dimensional Plane

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Spontaneous and Scientific Concepts in Mathematics: A Vygotskian Approach

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We are but a year from the centennial celebrations of Jean Piaget and Lev Vygotsky. These two men never met, but their work is linked in a variety of ways. Vygotsky translated some of Piaget's early writings into Russian, and Piaget was asked to write a commentary about Vygotsky's Thought and Language, thirty years after the Russian psychologist's death. Among students of mathematics education, Piaget's influence has been far greater than Vygotsky's influence. Recently, with increasing focus on the sociohistorical contexts in which mathematics is practiced (Tymoczko, 1986, Davis & Hersh, 1981), Vygotsky's sociocultural theories have come to interest educators and psychologists.

A central assumption of sociocultural theory relates to the nature of thought. While Piagetians stress formal operations as the highest stage of thinking, Vygotskians include formal logic, domain-specific modes of thought and dialectics in their characterizations of thought. The Piagetian approach reflects the powerful role of formalism in mid-century science, including the impact of the group of French mathematicians, Bourbaki. Their work stimulated others, including Piaget, to look for logically consistent modes of thinking and for universal laws. A central aspect of such an approach is the notion of invariant stages. In contrast, sociocultural thinkers stress the co-existence of different forms of reasoning within a community of learners, from which an individual appropriates different forms and processes in particular learning contexts.

Another difference between these two frameworks is the way the developmental origins of thought are conceptualized by Vygotsky and by Piaget. As Paul Cobb recently wrote, ". . . sociocultural theorists typically link activity to participation in culturally organized practices, whereas constructivists give priority to individual students' sensory- motor and conceptual activity" (1994, p 14.) This is a clear and useful distinction, but in Cobb's further analysis, he limits the concept of what is social by maintaining a hard boundary between the individual and the community. This distinction is reformulated by Vygotskians, who see a dialectical unity between the social and the individual. The boundary is porous, shifting, and re-experienced in a variety of ways. As a woman, this unity starts for me in prenatal life, where the interdependence of the mother and child precedes his/her separation and individuation.

In this presentation I will first outline some of the central themes of sociocultural theory. Then I will focus on the distinctions between *spontaneous* or everyday concepts and *scientific* or systematic concepts, and the way in which these concepts are woven together in the course of development. The Polish psychologist, Anna Sierpinska (1993) applied Vygotsky's theory of concept development to childrens' mastery of mathematical concepts. I will rely on her work in this talk as well as on the writings of V. V. Davydov (1972/1990, 1995), and on Jean Schmittau's (1993, 1994) comparative studies of American and Russian childrens' mathematical activities. Lastly, I will speak about learning as a collaborative process.

Sociocultural theory

Central to Vygotsky's study of thinking is his emphasis upon the genetic method, a focus that he shared with Piaget:

> We need to concentrate not on the *product* of development but on the very *process* by which higher forms are established. To encompass in research the process of a given thing's development in all its phases and changes—from birth to death—fundamentally means to discover its nature, its essence, for "it is only in movement that a body shows what it is." (Vygotsky, 1978, p.64)

According to this perspective, learning and development take place in meaningful, socially and culturally patterned contexts, and they need to be studied developmentally over both long time stretches and microgenetically. The best-known microgenetic studies by sociocultural researchers are those conducted by James Wertsch and his collaborators (Wertsch & Minick, 1990). In these studies of dyadic interactions, the investigator examines how a skill or strategy is first acquired at the interpersonal level, eventually to be incorporated into the novice's intra-personal repertoire.

Vygotsky situates learning within relationships among mutually interdependent individuals. In describing the process of language socialization from a Vygotskian point of view, we (John-

Steiner & Tatter, 1983) wrote

> From birth the social forms of child-caretaker interactions, the tools used by humans in society to manipulate the environment, the culturally institutionalized patterns of social relations, and language, operating together as a socio-semiotic system are used by the child in cooperation with adults to organize behavior, perception, memory and complex mental processes. (1983, p.83)

In this description, we suggest that human tools (as well as the many forms of interdependence) exercise specific roles during socialization. External tools are used in productive labor and in the regulation of environment/persons interactions. Psychological tools are less direct. Kozulin (1990) provides clocks as an example of such tools. Historically, the measurement of time was first linked to natural processes, such as the movement of sand in a hourglass. With the invention of mechanical and later digital clocks, the measurement of the passage of time became mediated. "In order for an individual to read a watch, the whole system of symbols such as digits, language abbreviations, positions on the screen, etc. have to be learned" (Kozulin, 1990 p.135)

Time-related psychological activities in contemporary technological societies are mediated by clocks, which become part of the human semiotic system. While physical tools are directed toward the external world, psychological tools have as their aim the mediation of activity and the representation of the meaning of such activity. For instance, the reporting, remembering and calculating of the passage of time with the help of clocks. Psychological tools are socially constructed, and individuals have access to these tools as participating members of their social world in which these tools have been made and shared.

In a recent edited book on "distributed cognition" (Salomon, 1993), Roy D. Pea applies some of these Vygotskian notions concerning psychological tools to basic mathematical reasoning. He mentions research on street candy selling by Brazilian children (Nunes, Schliemann, and Carraher, 1993); Scribner's studies of dairy workers, (1985); Jean Lave and her colleague's (1984) findings of grocery shopping, among others. Such studies highlight how people engaged in intelligent activities use resources to arrange the environment conveniently and thus "achieve less mental effort if necessary" (Pea, 1993, p.63.) Pea provides an additional detailed example of the role of socially developed tools which become part of computational activity

> An example of distributed intelligence comes from the PBS television show "Square One" on mathematics for children. A forest ranger is being interviewed. Each year she measures the diameters of trees in the forest to estimate the amount of lumber contained in a plot of land. (Pea, 1993, p. 69)

With a conventional measuring tape she has to remember the formula which relates circumference and diameter, and then she has to carry out some calculations.

> But something different has been invented. A new measuring tape which I call a special-purpose "direct calculation" tape for tree-diameter measurement. The numbers are scaled so that the algorithm for these calculations is built into the tape. (Pea, 1993, p.70)

Pea argues that "through processes of design and invention, we load intelligence into both physical, designed artifacts and representational objects such as diagrams, models, and plans" Pea, 1993). In providing contemporary examples of the roles of psychological tools in problem solving, Pea contributes to a rethinking of some of Vygotsky's notions concerning mediated cognition. Vygotsky emphasized language in most of his analyses of development. Similarly, contemporary sociocultural theorists have neglected some non-verbal psychological tools such as diagrams, by focusing solely on words. I have argued recently (John-Steiner, 1995) in favor of cognitive pluralism, the notion "that historically developed mediational means constitute an ensemble of psychological tools" (John-Steiner, 1995, p.3). Vygotsky (1981) himself listed a number of these:

> The following can serve as examples of psychological tools and their complex systems: language; various systems of counting; mnemonic techniques; algebraic symbol systems; works of art; writing; schemes, diagrams, maps and mechanical drawings; all sorts of conventional signs; and so on. (p.137)

But this is not a well-known quote. Vygotsky's primary focus on language may have contributed to limiting his impact on Western mathematics education. This situation is changing, in part, because of the great interest of educators in computers and in the graphing calculators used in calculus classes. Vygotskian theory, with its emphasis on psychological tools and mediated action, provides a theoretical framework for the role of these artifacts in cognitive activity. Roy Pea suggests that these resources have contributed to the transformation of the objectives and timing of the entire course of mathematics education (National Council of Teachers of Mathematics, 1991). For example,

> in K-4 mathematics, a focus on long-division operations and paper-and-pencil fraction computations has been diminished, the availability of calculators is assumed, and attention has shifted to estimation activities and a focus on the meaning of operations and selection of appropriate calculation methods (Pea, 1993, pp. 72-73).

The role of artifacts in cognitive operations, which is now taken for granted, has contributed to a rethinking of what some have called the "cranial storage metaphor" of cognition (Rogoff, Baker-Sennett, & Matusov, 1993, p.3). Learning is increasingly conceptualized by sociocultural theorists as distributed, interactive, contextual, and the result of the learners' participation in a community of practice. One of Vygotsky's concepts, that of the **zone of proximal development** (ZPD), is particularly relevant to this social and participatory view of learning. The following definition of ZPD is quoted from Mind in Society, an edited book of Vygotsky's writings:

> (it is) the distance between the actual developmental level as determined through independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers (Vygotsky, 1978, p.86).

Ann Brown and her collaborators (1992,1993) have expanded this concept to suggest that the active agents within ZPD "can include people, adults and children, with various degrees of expertise, but it can also include artifacts such as books, videos, wall displays, scientific equipment and a computer environment intended to support intentional learning" (1993, p.191).

The role of complex artifacts in cognitive operations is now taken for granted, particularly by educators interested in distributed cognition. There are restrictions, however, in the availability of these technological systems to learners in economically underdeveloped countries. It is important that in our theories of learning we do not universalize learning strategies beyond the contexts in which they were acquired and practiced. It is here that research in diverse cultural settings, (for instance, John-Steiner & Panofsky, 1992; Nunes, Schliemann & Carraher, 1993;

Rogoff, 1994; and Saxe, 1991, 1994), serves as a useful counterweight to universalistic approaches.

It is not possible to summarize all sociocultural concepts in this presentation. My purpose it to choose those concepts from the sociocultural framework which are relevant to the psychology of mathematics education. Among these are psychological tools and their role in mediated activity, the zone of proximal development, and my own notion of cognitive pluralism. There are two more concepts that need to be reiterated before I move on to the distinction between every-day and scientific concepts. One of these is Vygotsky's notion of the heterogeneity of thought, as manifested in the coexistence of genetically different forms: "[D]ifferent genetic forms coexist, just as in the earth's core the deposits of quite different geological epochs coexist" (Vygotsky, 1956, quoted in Wertsch, 1991, p. 99). When an adolescent or an adult confronts a new problem or situation, he or she usually has difficulty in differentiating significant features from insignificant ones. At the same time, while the novice operates at a very simple level in the new problem space, the same person is an expert in many other contexts. The co-existence of developmentally varying forms within the same individual is an important application of Vygotsky's geological metaphor. His notion is also relevant to diverse forms of thinking in mathematics, both between and within individuals; for instance, consider the varying and interesting roles of analytic and analogue approaches in mathematical problem solving (Davis & Hersh, 1982). The differential appropriation of mathematical approaches, such as visual-geometric, (or analogic) as well as formal and analytical approaches contribute to pluralism in mathematical thinking.

Lastly, Vygotskians claim that the mastery of concepts is not achieved through individual mastery, but is the outcome of socially embedded and socially facilitated activity. The centrality of this notion within the Vygotskian framework is effectively captured by Thomas Bidell who wrote in 1992: "According to this conception, dimensions of reality such as the social and the personal are not separate and self-contained but have a shared existence as differing tendencies united within real developing systems" (p.308).

One of the ways to conceptualize the unification of processes which many think of as separate, is through activity: participation in communities of practice (Lave & Wenger, 1991) or within communities of learners (Rogoff, 1994). The social and interdependent nature of learning and development is particularly striking in the acquisition of school-taught, scientific concepts.

Spontaneous and scientific concepts

Scientific concepts, or what some authors have referred to as systematic or theoretical concepts "originate in the highly structured and specialized activity of classroom instruction and are characterized by hierarchical, logical organization. The concepts themselves do not necessarily relate to scientific issues—they may represent historical, linguistic, practical knowledge—but their organization is "scientific" in the sense of formal, logical and decontextualized structures. Every-day (or spontaneous) concepts, on the other hand, emerge spontaneously from the child's own reflections on immediate, everyday experiences; they are experientially rich but unsystematic and highly conceptual" (Kozulin, 1990, p.168). Vygotsky's terminology in describing concepts was partly borrowed from Piaget, who wrote of spontaneous and non-spontaneous concepts. In Kozulin's definition, "spontaneous" remains individually constructed, reflecting a Piagetian core meaning. Other writers, for instance, Van der Veer and Valsiner (1991) bring a different interpretation to the word. They write, "by spontaneous concepts he [Vygotsky] meant concepts that are acquired by the child outside of the context of explicit instruction. In themselves these concepts are mostly taken from adults, but they have never been introduced to the child in a systematic fashion and no attempts have been made to connect them with other related concepts" (p.270).

A frequent example illustrating spontaneous concepts is drawn from the domain of family relationships: a child knows who his brother is, but has difficulty with the puzzle, "Who is my father's son who is not my brother?" (Kozulin, 1990, p.169). Clearly, the child has a variety of everyday experiences with a brother. In English we don't know from the word used whether the brother is older or younger than the speaker. In Hungarian, my native language, the distinction is provided by choice of noun: "báttyám" for older brother, "öccsém" for younger brother. Thus, the specific term or terms for "brother" are provided by the speech community. The child does not make them up. In the course of "constructing" everyday concepts, the child combines the sense of direct experiences, which are rich in personal meaning, with words which are socially constructed and transmitted. Thus, both Kozulin's sense of personal and Van der Veer and Valsiner's socially mediated interpretations of Vygotsky's notions are relevant to the unpacking of the meaning of "spontaneous" concepts. Although the characterization of scientific and spontaneous concepts has been a subject of debate among sociocultural researchers (see Gee's recent manuscript, 1995), the use of these concepts has been productive in research and in the development of new teaching approaches.

In a Dutch study of young children's semiotic activity (van Oers 1994a), Vygotskian approaches were applied in an imaginative way. The children who participated were asked to play with appealing toys set up in different corners of their classrooms. One group of children, who were playing with a toy train, were told that they had to communicate about their train to pupils in another school. In order to do so, the children had to look at different parts of the track and to prepare for reassembly by trying to remember what the outline of the circuit looked like, etc. The teachers encouraged the children to develop their own diagrams, narratives, and related notational systems in preparation for the communication with the children in the other school, an activity which did indeed take place. The teachers scaffold the system of analyzing the track by the questions they pose and by the assistance they provide with the notational system. In a related study (van Oers, 1994b), kindergarten aged children's dramatic play as "shoemakers" and "costumers" was observed and their interactions were tape recorded. The young learner's activities, which included measuring, counting, and the use of notational systems, illustrated the use of mathematical concepts. In these studies, the concepts of addition, division, and schematic representations are embedded in children's play activities at an early age.

Sociocultural researchers, such as van Oers, do not view the acquisition of mathematical concepts as following a single, universal time table. They suggest that the effective teaching of these concepts is based on theoretical and pedagogical analyses and on imaginatively arranged contexts and tasks. These concepts can be introduced to young learners if scaffolded by carefully designed social experiences.

The Russian psychologist V.V. Davydov (1972/1990) applied Vygotsky's distinctions between everyday and scientific concepts to mathematics. He suggested that everyday concepts are developed by empirical abstractions by comparing features of objects, and phenomena, at the level of appearance. Empirical abstractions may be false. For instance, young childrens' belief that when two objects fall, the heavier of the objects may fall more rapidly than the lighter one. (Howard Gardner uses this as an example of the "unschooled mind.") An example that Schmittau (1993) mentions in explaining Davydov's ideas is that of the diurnal cycle, "in which the perceptual illusion of a sun rising in the east and setting in the west belies the scientific understanding of a phenomenon produced by the rotation of the earth on its axis" (Schmittau, 1993, p.30). Davydov's central claim is that scientific concepts require a theoretical abstraction which reveals their essence. In describing what is meant by "the essence" of a mathematical concept, Schmittau considers Aristotle's distinction between the two ways in which polygons can be categorized. In the first of these, polygons are grouped according to the number of their sides. This method of formal generalization does not reveal the same generality as achieved by genetic analysis in which the triangle is seen as primary "since all other polygons can be generated from and decomposed into triangles (Schmittau, 1993, p.31.) The latter method, then, reveals an essential aspect of a mathematical object.

One of Davydov's students, the Siberian mathematics educator, L. K. Maksimov, developed some teaching techniques based on Vygotskian analyses (1993). He was interested in elementary students' mastery of the order of mathematical operations. He suggests that children discover the problems involved in such operations when they get different answers for the value of the same mathematical operations. Rather than relying on formulaic resolutions of the order problem, Maksimov provides them with new ways of representing their actions, in this case a "mathematical tree". The method also includes group problem solving approaches, where different students assume responsibility for different operations such as multiplication or addition. These teaching approaches highlight the importance of theoretical generalizations as well as purposeful action, the role of contradictions, the usefulness of cooperative learning, and the importance of new notational systems. Jean Schmittau (1994), who has been instrumental in documenting the effectiveness of these Vygotskian teaching methods in mathematics, compared Russian and American students' understanding of multiplication, including their varied ways of representing and explaining this mathematical operation. While American students relied on a cardinal operational framework, and used prototypic examples with small numbers, Russian children relied on mathematical trees, used more interesting numbers including fractions, and were able to solve new problems. "... every Russian subject, including the children from the fourth and fifth grade, who had never been introduced to binomial multiplication, was able to obtain the product of two binomials and explain in what sense it represented multiplication" (p.17). As part of the Vygotskian approach, Russian children engage is scaffolded problem-solving before they are given algorithms. "[They] will be given problems such as 4/5 of 2/3 to solve by carrying out the requisite actions during the fifth grade, with the algorithm for the multiplication of such fraction is delayed until sixth grade" (Schmittau, 1993, p.31).

Before children achieve true conceptual thinking, their concepts take their meaning from the perceptual, functional, and contextual aspects of their reference. The challenge of effective teaching from this point of view is to assist children in acquiring conceptual connections and to discover underlying stability in the face of surface variations. Maksimov (1993) uses mathematical trees as scaffolding devices, Sierpinska (1993) poses challenging problems to children and adolescents and assists them during lengthy interviews with the development of complex arguments. One of the problems she posed dealt with the "question whether the set of all natural numbers can be regarded as having as many elements as the set of all even numbers" (p.100). She found that adolescents when confronting such theoretical problems are frustrated and argumentative, and she quotes Vygotsky who wrote: "the age of adolescence is not the age of completion of the development of thought; it is the age of its crisis and maturation" (p.104). The mastery of scientific concepts requires scaffolding on the part of the experienced thinker who pays close attention to learners' productive errors, to their intellectual crises and disturbances, and to the ways in which they synthesize their varied experiences.

Complementarity and Cognitive Pluralism

In the early decades of this century, Vygotsky stressed the **intergenerational** scaffolding of the acquisition of knowledge while contemporary researchers have documented the value of **peer** collaboration (Forman & Cazden, 1985; Palincsar & Brown, 1984; Lambdin, 1993; and van Oers, 1994b). These researchers have shown how, in joint activities, learners can solve problems which they are unable to solve alone. Cooperative problem-solving provides the opportunity to verbalize thoughts, to articulate different perspectives which enrich each individual's understanding Although sociocultural theorists are committed to the exploration of joint activities, our theoretical analyses have not kept up with the rapidly developing practices of collaboration

It is my intention to discuss the development of such a theory in greater detail during my presentation The developing ideas of my collaborators (Michele Minnis, Bob Weber, Carolyn Kennedy, and Teresa Meehan) and I include the consideration of collaborative roles, values, complementarity and cognitive pluralism We view complementarity as central to shared activities, whether such complementarity is a temporary one, as in Maksimov's mathematics classes where different students are given responsibilities for addition, or multiplication tasks, or in situations which require a more lasting divisions of labor We are interested in linking complementarity, division of labor, and group interactional processes with the sustained use of diverse semiotic means in the course of collaboration Some individuals rely on diagrammatic representations of ideas while thinking together and others write down the consequences of thoughtful interactions Words, drawings, scientific diagrams, mathematical notations, and musical notes are part of the psychological tools with which knowledge is transmitted, explored and transformed In examining the acquisition of mathematical knowledge from a sociocultural perspective, we are engaged in exploring, expanding and reformulating Vygotsky's generative notions I am looking forward to participating with you through dialogue, controversy and the development of new syntheses in the varied contexts of psychological and mathematical discourse

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Some Concerns about Bringing Everyday Mathematics to Mathematics Education

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The Psychology of Mathematics Education has made considerable progress in the last 20 years. Many issues that had long been ignored have surfaced and are now being followed with interest. There is now a greater awareness that mathematical knowledge is constructed by the child. We know that meaning and representations are important and, certainly, more informative about processes of learning than simple measures of performance; skill and competence have not ceased to be important, but they are only part of a larger picture. Researchers have increasingly examined the epistemological obstacles and the constraints of specific symbolic systems. The very notion of "symbol" has become broader and richer; whereas several decades ago many mathematics educators would have thought primarily about notational systems, it is now increasingly recognized that symbol systems and symbolic representations must also include the use of language, graphs, diagrams, and even kinesthetic expression. We now know that many of the key issues in the elaboration of mathematical understanding concern the establishment of interrelations among representations in diverse symbolic systems. It has also become increasingly clear that mathematical knowledge has important social, cultural, and historical dimensions that need to be understood. Our own work in Everyday Mathematics is but one reflection of this trend.

A consensus is comfortable. It is nice to know, for example, that most math educators consider themselves to be constructivists. I too consider myself to be a constructivist. But aren't we coming too quickly to an agreement about the big issues, an agreement about general theoretical principles, before we have given enough attention to research. We hear a lot of discussion today about the importance of "meaning", for example. The concept of meaning is sometimes used as if it were basically well understood. But do we know, for instance, what meaning really is and how it is constructed? Similarly, we now recognize that "contexts" are important in learning. But what do we really know about contexts? We generally agree that children's previous knowledge and everyday experiences are important for both learning and instruction. But how do we get beyond such general formulations, which sometimes sound like slogans, to more precise formulations?

My own view has been that empirical research can bring us closer to understand how people think about, use, and experience mathematical ideas in particular contexts. I also believe that it is through this understanding that we can find better ways to provide opportunities for children to develop mathematical knowledge in schools. In what follows, I will first provide an overall view of the findings regarding how children and adults come to develop and use mathematical knowledge in everyday situations, focusing on some of the key issues raised by studies comparing the mathematics people use in their lives with the mathematics they learn or should learn in schools. In this overview I would like to share with you some of the insights we have gained from research into everyday thinking. After reviewing the research on everyday mathematics, I would like to look at some recent examples of people learning during activities designed as part of research projects I carried out with others. My intent is to show you how and why such observations have forced us to reconsider some of the ideas that we initially held about

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Everyday Mathematics and its relation to Mathematics Education. Then I will discuss implications that these and other studies seem to hold for the teaching of mathematics.

What have we learned about mathematics in and out of schools?

Developmental psychologists have long recognized the importance of children's experiences and interactions in the world for the development of logico-mathematical knowledge. Understanding that the number of objects in a set is not altered if their spatial configuration changes, that a part is always smaller that the whole, or that if an object A is larger than B and B is larger than C, then, necessarily, A is larger than C, are basic mathematical concepts acquired independently of school instruction, through children's actions and reflections upon the consequences of their actions in objects and situations in the world (Piaget, 1967).

The importance of previous knowledge in the development of new understandings is stressed in developmental analysis such as Piaget's and Vygotsky's. In his theory of cognitive development Piaget proposes that the development of logico-mathematical concepts takes place when children face situations they consider problematic and to which they first try, unsuccessfully, to apply previous available knowledge. In Piaget's words, children try to assimilate the new situation using their previous strategies. When this fails, they try new approaches or, in other words, they accommodate developing new strategies to take into account peculiarities of the situations. In this process, for a situation to be problematic, it must be related to children's previous understandings (Piaget, 1967). In Vygotsky's analysis of the development of scientific concepts, he stresses the idea that "scientific concepts can only be born in children's minds from previous elementary and inferior generalizations" (Vygotsky, 1985, p. 222).

More recently, studies of everyday cognition and analyses of use of mathematics in out of school contexts contributed to increase awareness among mathematics educators that mathematical learning can take place outside of instructional contexts and that children, when they enter school, already understand many basic mathematical concepts. Specific socio-cultural activities such as buying and selling promote the development of mathematical knowledge that was previously thought to be only acquired through formal instruction. Strategies for solving arithmetical operations, the properties of the decimal system, or understanding and solution of proportionality problems are examples of mathematical knowledge displayed by groups with restricted school experience. Besides arithmetic (T. N. Carraher, Carraher, & Schliemann, 1982, 1985; Saxe, 1990; Lave, 1977), contents as varied as measurement (T. N. Carraher, 1986; Gay & Cole, 1967; Saraswathi, 1988, 1989; Saxe & Moylan, 1982; Ueno & Saito, 1994), geometry (Abreu & Carraher, 1989; Acioly, 1994; Gerdes, 1986, 1988; Harris, 1987, 1988; Millroy, 1992; Zaslavsky, 1973), and probability (Schliemann & Acioly, 1989), have been found to be used or understood in everyday settings by children or adults with very little access to school instruction.

Although starting from different methodological traditions and theoretical frameworks, studies of conceptual development (for instance, Gelman, 1979; Hughes, 1986; Resnick, 1986) share with everyday mathematics studies the conclusion that, independently from school instruction, children develop a fairly rich body of mathematical knowledge through their everyday experiences in the world. Another common finding concerns the role specific situations and contexts play in what aspects of logico-mathematical relations will be focused upon and on how knowledge is displayed and represented. Developmental psychology studies by Dias and Harris (1988), Donaldson (1978), Light, Buckingham, and Robbins (1979), McGarrigle and Donaldson (1974), among others, have long demonstrate the importance of the contexts where cognitive tasks take place. They show that children who fail in formal syllogistic reasoning problems or in the traditional Piagetian conservation, class inclusion, or perspective taking tasks, demonstrate logical reasoning when the questions are set in more natural and meaningful contexts.

Contexts also play a central role in mathematical achievement: When arithmetical problems are solved at work by young street vendors, answers are usually correct, while in school like situations correct answers are much rarer (T. N. Carraher et al., 1982, 1985). Such findings are, without any doubt, extremely relevant for how children from poorer communities are looked at when they enter school and have difficulties in coping with routines they have to memorize without understanding, or with numerical symbols that are not related to the way they represent and work out mathematical relations.

Acknowledging development and use of logico-mathematical reasoning in children's everyday strategies is a first step towards promoting opportunities for progress and learning. But then, when at school, how can children's previous knowledge and experiences be considered as the basis for learning school mathematics? Should we simply replicate everyday tasks in the classroom and try to guide children in deeper exploring their implicit mathematical relations? Can we find everyday tasks that would fit all or most of the contents we want to be part of the mathematics curriculum? And, if so, once an everyday task is transplanted into the classroom, are we dealing with the same task? Can we expect that children will be as involved in mathematical analysis as they are when they search for solution to problems in everyday life situations? These are issues mathematics teachers and researchers are facing (see, for instance, Civil, 1985) and they demand closer analysis of the characteristics of everyday mathematics in order to be dealt with. In what follows I will look at the strengths and limitations of everyday mathematics in regard to three aspects I consider as mostly relevant for Mathematics Education: (a) the question of meaning in arithmetical problem solving, (b) the question of generalization and transfer of mathematical knowledge across different contexts and situations, and (c) the question of referents for mathematical symbols.

Problem solving strategies and meaning in and out of schools

Street vendors develop, through their everyday working activities, a basic understanding about the properties of our numerical system (see Nunes, Schliemann, & Carraher, 1993, for a detailed analysis on street vendors strategies and understandings). Their failure in school arithmetic is not due to deficits in their ability to reason mathematically, but seems to be rather related to failure in adopting the particular symbolic systems and conventions taught by schools. As shown by T. N. Carraher, Carraher, & Schliemann (1987) in their analysis of the differences in an individual's problem solving strategies across situations, the quality and effectiveness of mathematical reasoning seems to be closely related to the nature of the representations being used: Out of schools people rely on oral mathematics; in school written procedures are the rule.

The contrast between absurd results in school contexts versus correct results in work contexts is a common feature of data by T. N. Carraher et al. (1985), Grando (1988), Lave (1977), Reed and Lave, 1979), Schliemann (1985), and Schliemann and Nunes (1990). These and other studies are rather consistent in showing that arithmetic algorithms traditionally taught in schools to provide students with powerful general procedures are not always helping them outside of classroom settings and that strategies developed by people in practical settings appear to be more efficient than school algorithms.

As is often the case, students learn algorithms in schools as steps to be followed automatically, without considering the implicit mathematical properties and relations that allow their construction. Algorithms are meant to be efficient and quick, thus reducing the mental load involved in working out solutions. But speed and efficiency are school algorithms' strength as well as their limitation since focus on the use of rules often lead to answers without considering the specific aspects of the situation involved. Moreover, focus on rules to be learned without considering the implicit mathematical relations that allow their construction usually leads to wrong steps when memory fails.

In contrast to school algorithms, the strategies developed by individuals as tools to solve problems at work are characterized by constant reference to the physical quantities involved, throughout the necessary computational steps, and by reliance on transformations that take into account the properties of the decimal system.

It thus seems to be the case that everyday mathematics, although based on less powerful oral strategies reveals understanding of mathematical principles and almost always lead to correct, meaningful answers. What about the limitations of everyday mathematics? Is everyday mathematical knowledge context specific? Is it "concrete"? Can we hope that all complex facets of mathematical concepts be dealt with in everyday settings?

The problem of transfer: What a difference a slight change in context makes

Several years ago Nadja Acioly and I carried out research regarding Brazilian bookies that work in what is known as the Animal's Game.

Among many other mathematical features, the game involves placing bets on diverse permutations of numbers (see Schliemann & Acioly, 1989, for a detailed description of the mathematics involved in the game). A permutation is of course an arrangement (or an arranging) of elements according to a particular order. We adapted permutation conditions routinely used in the Animal Game to see whether the bookies would generalize for cases with elements of diverse natures. For example, we thought that the set of letters,{c, a, s, a}, which spell the word "house" in Portuguese, would be seen as analogous to the set of numbers {1, 2, 3, 2}. Mathematically, the problems are isomorphic in the sense that we can directly match properties of one version to properties of the other. However, what from our point of view appeared to be a trivial alteration (letters were used in the place of numbers) turned out to be not so trivial for some of the bookies, as the following transcript makes clear:

> Examiner: I want you to find out in how many different ways you can arrange the letters in the word CASA (show word written on paper) without leaving any letters out and without putting in any other letter.
> Subject (who has just given the permutations for a set of numbers): This one is even harder (than with numbers) because I can't read.
> E: But you don't have to read. I want you to tell me about how many different ways you can change the position of these letters.
> S: I can't to this.
> E: What if you try to do it as in the Animal's Game?
> S: This is very hard because reading is more difficult than working with numbers. I know how to do a few calculations but I don't know how to read.
> E: What if you make believe that "C" is a number like "1", the "A" a number like "2", the "S" is number "3", and this "A" is number "2". Couldn't you do it?
> S: No, because one thing is different from the other.
> (From Schliemann & Acioly, 1989, p. 206).

The above dialogue puzzled us. How could it make a difference whether numbers or letters were used? When we looked across the bookies, to see what factors would predict how they responded to such a task, we found an intriguing relation to their level of schooling. Only bookies with less than one year of schooling failed to respond correctly to the letter version. Cross-cultural studies of logical reasoning (Luria, 1976; Scribner, 1986) had already shown that as little as one year of formal school makes a difference in how people deal with logical reasoning tasks. But why is it so? Failure to transfer is not specific to illiterate people. In fact, in a follow-up analysis (Schliemann 1988) we found that many college students were unsuccessful in noting the similarity between the letters version and what they had learned in school about permutations. It thus seems that failure to transfer among schooled or unschooled people is not simply a direct consequence of their levels of schooling but is also related to the way they use their mathematical procedures. Everyday procedures used by lottery bookies to determine the number of permutations in a set of digits does not require working out the permutations but only to know how many arrangements are to be found for a given set of numbers. Since tables listing the number of arrangements for each possible set are available, they only have to memorize the correct answers, without understanding the structure of task. In a way, they are acting as school children when they memorize algorithmic procedures.

But would unschooled people show transfer of problem solving strategies if they have a deeper understanding of their procedures? Schliemann and Nunes (1990) analyzed transfer of strategies for solving proportionality problems among the participants of a community of fishermen in Northeastern Brazil. In that community, the fishermen's everyday experience requires them to repeatedly compute the price of a certain number of items to be sold, given the price of one item. Their understanding of proportional relations however appears to go beyond this repetitive procedure: they are able to invert their procedures and compute the price of one item, given the price of more than one, thus showing flexibility, one of the characteristics of conceptual knowledge (Hatano, 1982). In this case, regardless of their levels of schooling, fishermen were also able to transfer their procedures in order to solve proportionality problems relating quantity of processed to unprocessed seafood, a relationship they used to refer to in their activities but where no computing problems ever occurred.

More recently Veronica Magalhães and I (see Schliemann and Magalhães, 1990 and Schliemann & Carraher, 1992) provided a further analysis of transfer of strategies for solving proportionality problems among unschooled people. We asked female cooks just enrolled in an adult literacy class, or enrolled one year before they participated in the study, to solve series of missing value proportionality problems. The problems involved quantities in two contexts that were part of the subjects everyday experience (a sales transaction context and a cooking context), and an unknown context of a pharmaceutical mixture of ingredients. The cooks in one group were first given recipe problems, followed by problems about prices, then by a repetition of the recipe problems, and finally the medicine problems. A second group encountered the problems in the order price, recipe, medicine, and recipe. Finally, a third group received the problems in the order medicine, price, medicine, and recipe problems. For the first context encountered, while price problems were nearly always precisely solved, only occasionally precise answers were given for recipe and medicine problems: about half of the solutions for recipe problems were estimates of quantities to be used and justifications tended to be rather informally expressed, as in "I think that's enough" or "That's how I do it"; for medicine problems roughly one-half of the answers were outright wrong answers that often appeared to be obtained by guessing or by performing a meaningless operation upon the given quantities. Results for recipe problems in the second presentation context were strikingly different from the first: the percentage of correct responses jumped from 18% to 61%. For medicine problems presented after money problems, but before recipe problems, the percentage of correct answers remained low (27%). However, after solving money problems and recipe problems, the improvement for medicine problems was quite remarkable (62% of correct answers).

As a whole, we can conclude that, although school experience may play a role in the way people face problems in unknown contexts, mathematical knowledge developed in everyday contexts is flexible and general. Strategies developed to solve problems in an specific context can be applied to other contexts, provided that the relations between the quantities in the target context are know by the subject as being related in the same manner as the quantities in the initial context are.

Let's now discuss the consequences of working with numbers to represent specific measured quantities.

"Concrete" referents: Who needs them?

Out of school we use mathematics to reflect upon and to decide about situations involving quantities or measures of objects and it is not natural to refer to pure numbers or to explore the relations between numbers by themselves. Young children (see Hughes, 1986), as well as adolescents and adults with restricted school experience (Schliemann, Santos, & Canuto, 1992) typically do not understand what one means by "How much is 4 plus 2?", or "How much is 27 plus 19?". If, however, the questions are rephrased as "There are 4 bricks in this box. I'm going to put in it 2 more bricks. How many bricks there will be in the box?", or as "If you had 27 cruzeiros and then someone were to give you 19 cruzeiros more, how many cruzeiros would you have at the end?" they are able to work out correct solutions, even though the physical objects (bricks, coins, or money bills) are not present.

One example of continuous use of referents in street mathematics is the street sellers procedure to solve problems that are identified in schools as multiplication problems (see T. N. Carraher et al., 1982, 1985; Nunes et al., 1993). When computing the price of a certain amount of the items they sell, starting from the price of one item, street sellers usually perform successive additions of the price of one item, as many times as the number of items to be sold. This falls into what Vergnaud (1982) describes as the scalar approach for solving missing value proportionality problems. In this case each variable remains independent of the other and parallel transformations that maintain the proportional relationship are carried out on both of them. Another way to deal with proportionality problems is the functional approach which focus on the ratio between the starting values of the two variables and applies this ratio to find the missing value in the final pair (Vergnaud, 1982).

Schliemann and Carraher's (1992) provide examples of the differences between street sellers and school children's strategies for solving proportionality problems. Underlying street sellers' strategy lies an understanding of the relationship between the two variables, number of items to be sold and price to be paid, as proportionally related. However, although revealing preservation of meaning and understanding of proportional relations, there seems to be limits to street sellers problem solving ability: When problems are set in such a way that the relation between price and number of items (the functional relation) is easier to be worked out than the relation between the starting and the ending quantities (the scalar relation), school children most often focus on the functional relation while street sellers continue to use the scalar strategy, even when this requires more cumbersome computations as in the following example:

Subject: Flávio, 13 years old ice-cream vendor for two and a half years.
E: 3 pens cost 9 cruzeiros. With 21 cruzeiros, how many pens can you buy?
S: 3 pens is 9 cruzeiros, 6 is 18. 1 pen is 3 cruzeiros. 18 to 21 is 3. 6 plus 1 is 7. 7 pens cost 21 cruzeiros.
(From Schliemann & Carraher, 1992, p. 68).

If, however, the starting amount is larger than the ending amount, street sellers' strategies (see Carlos following example) are in clear disadvantage when compared with school children's ability to focus on the functional relation (as in Edson's example below):

Subject: Carlos, a 13 year old chocolate vendor.
E: 21 chocolates cost 9 cruzeiros. How many chocolates can I buy with 3 cruzeiros?
S: (after a pause and after counting his fingers) 9 chocolates.
E: While you were solving it, you were thinking. Tell me now what were you thinking about.
C: 9 plus 3 makes 12, 21 minus 12 makes 9.

(From Schliemann & Carraher, 1992, p. 67).

Subject: Edson, 6th grader.
E: 9 erasers cost 3 cruzeiros. You want to buy 21 erasers. How many cruzeiros will you need?
S: This one is really meant to get me. If 9 erasers cost 3 cruzeiros, then 21 erasers cost (pause). This one looks like the other one. It is 7.
E: How did you find out?
S: Because the triple of 3 is 9, and the triple of 7 is 21. 3 times 3, 9, 3 times 7, 21.
(From Schliemann & Carraher, 1992, p. 68).

Another example of the limitations related to use of numbers with specific referents concerns the understanding of multiplication as a commutative operation. Failure to recognize the commutative law for multiplication among non-schooled subjects is described by Petitto and Ginsburg (1982) among Dioula tailors and cloth merchants. For addition and subtraction problems, their subjects could easily use associativity and commutativity while solving the problems and their answers were nearly always correct. For multiplication problems, however, they could solve a problem involving 100 x 6 by adding 100 six times, but did not accept that the same result would apply for the computation of 6 x 100. Later Schliemann, Araujo, Cassundé, Macedo, and Nicéas (1994) found that, compared to school children who use multiplication, young street sellers who use repeated additions to solve multiplication problems show a developmental delay in accepting the commutative law for multiplication. When computing the price of many items, given the price of one item, what makes sense to them is to add the number that refers to price as many times as the number denoting how many items are to be sold. Thus it seems inappropriate for many street sellers to consider multiplication on quantities as commutative, that is, to add the number of items as many times as the price of each one to find the total price, even when this represents a great economy in addition have to be performed.

However, despite their more powerful strategies for solving proportionality problems, school children show a relatively high number of errors for problems where the quantity of items is stated as a number larger than the number referring to price. As discussed by Schliemann and Carraher (1992), school children's difficulty lies in the loss of the referent for the result of the ratio between the two starting quantities. They compute the ratio but fail in identifying whether this denotes the price of one item or how many items you can buy with one monetary unit.

Our review of everyday mathematics consistently shows that meaning is brought into everyday problem solving activities through constant presence of specific referents. But exclusive use of numbers with such situational referents imposes limits to mathematical relations. This brings us to a conflicting situation: to ensure that mathematical problems are dealt with meaningfully, mathematical symbols should always be used in connection to the physical quantities they represent. But what about generality and focus on mathematical relations independently from specific physical referents? Shouldn't school mathematics, even though recognizing the roots of mathematical understanding as deeply related to physical referents, also aim at meaningful understanding of general mathematical relations? What would then be the proper way to deal with the problem of meaning in schools? Is it possible to develop activities where numbers and algorithms become meaningful objects to be reflected upon even though they are not restricted to links to specific everyday situations?

Can meaning be brought into school mathematical activities?

The examples and results reported above show that the same cultural and social environments that allow construction of mathematical knowledge also constrain and limit the kind of knowledge children and adults will come to develop. The analysis of the strengths and limitations of everyday mathematics lead to the question of how can we design better opportunities for children to develop mathematical knowledge that is wider than what they would develop outside of schools, but that preserves the focus on meaning found in everyday situations.

As described by Resnick (1987) school learning focuses on individual cognition, pure thought activities, symbol manipulation, and general principles, while out of school learning is characterized by shared cognition, tool manipulation, contextualized reasoning, and situation specific competencies. Within out of school contexts mathematical principles and properties are tools to achieve goals that have practical and social relevance for the individuals involved. We believe that this use of mathematics as a tool to achieve relevant goals is the main characteristic of everyday mathematics that should inspire the design of more appropriate school activities.

What follows is an example of an activity analyzed by D. Carraher and Schliemann (1991) in which children engage in discussion, use, and reflection about mathematical properties and symbols using their previous understandings and new discoveries as tools to play a software game that was extremely motivating for them. An important feature of the activity is that it seems to provide the proper environment for meaningful reflection about mathematical relations and symbolic representations even though the objects children deal with have no physical referents or material embodiments in the physical world.

We know that, from their everyday experience in the context of sharing or distributing things, even young children understand the idea that the remainder is the part of the original quantity that is left over. When computing long division results, however, the meaning of the remainder is often a puzzle for school children. They do not know what quantity the reminder represents and they fail to understand how the remainder relates to the decimal representation when divisions are performed in a calculator.

In the transcription to follow, Pedro and Taís have been working with Divide and Conquer, a software designed by David Carraher (D. Carraher, 1991). The software is structured around the division identity, that is, division with a remainder. (The basic idea behind division with a remainder can be simply stated: any natural number, A, can be expressed as an integral multiple of any non-negative integer, B, added to some other integer, R, such that $A = B*Q + R$, where Q is a partial quotient. For instance, the number 37 can be expressed as an integral number of 8's, namely, four 8's plus the remainder, 5.) The students' involvement in the task is triggered by the puzzle they have to solve, namely to break a code determining which of the 10 digits from 0 to 9 stands for each one of 10 random letters displayed on the computer screen. To help them break the code they can ask the computer to perform divisions with a remainder. The results shown by the computer, however, are also partially coded according to the conventions of cryptarithmetic. The students must reflect upon the relations between the numbers displayed in order to find out solutions.

In the following excerpt, Pedro and Taís are coming to realize that division in the Divide and Conquer game does not always give the same result as division by a calculator. Let us listen to their discussion about the two types of division. They start with division with a remainder, working within the limits of everyday multiplication and division where nearly always only whole numbers are dealt with:

> (Pedro and Taís have just noted that 89 divided by 2 will yield 44 remainder 1 on the computer.)
> Interviewer: If I divide (89 by 2) on the calculator I get the same answer, right? (pointing to 44.5 on the calculator).

> Taís (shaking her head): No, I think that (on the computer) 1 was left over and here (pointing to the right part of the answer, that is, to .5) it's half of one.
>
> Interviewer: This point-five is half of one?... Do you think it's the same thing, Pedro?
>
> Pedro: (is) five half of one? Does one have a half?
>
> Interviewer: Explain it to him Taís.
>
> Taís: I thought like this. Here (in Divide and Conquer) it gave 44 with 1 left over and there (on the calculator) this point-five is one half. Forty four and one half and (the computer) had forty four remainder one.
>
> Pedro: Ah!. Now I get it. One half of 1 is one-half and that's equal to 5. (Pedro seems to be a uncertain about the meaning of the 5 in the present context but is willing to agree with Taís that it means one-half).
>
> Interviewer (moving to another example): And now, 80 divided by 7....gave 11 remainder 3 (on the computer). Now let's try it out on the calculator.
>
> Pedro (transposing the former example literally to the case at hand) It's going to be point, um, five.
>
> Interviewer: Why are you telling me it's going to be (eleven) point five?
>
> Pedro: Because half of three is one and a *half*.
>
> (Calculator shows 11.428571 as the answer).
>
> Pedro: This thing here (.428571) is half of three.
>
> Interviewer: Really?...Are we dividing in half or not? (...) Isn't the problem 80 divided by *seven*?
>
> Taís: I can't explain that.
>
> Pedro (persisting): It must be half of 3.
>
> Taís: No way! All this (.428571) can't be half of 3.
>
> Pedro: So...? I though that this here (.428571) taken three times would give 3, but that's not right.
>
> Second Interviewer (continuing): So where does this (.428571) come from?...
>
> Students: (no reply)
>
> First Interviewer: You are saying that that is a little piece of three, but what piece is it?
>
> Pedro: That 7 has got to have something to do with it.
>
> E: What do you think it has to do with it?
>
> Pedro: This here (.428571), seven times, gives us the 3.
>
> (From D. Carraher & Schliemann, 1991, pp. 28-29).

What is going on here? Is the problem meaningful for Pedro? It seems that we need to be careful about generalizing. Pedro stumbles at a number of points in the interview. For example, he seems not to understand much about the notation for the decimal fraction that corresponds to 3/7. That is, he does not seem comfortable with decimal notation. However, he clearly has some understanding and, despite this limitation, Pedro seems to have discovered that the fractional part of the quotient, *when multiplied by the divisor*, will return the value of the remainder of integer division, therefore relating division with a remainder to long division. He would certainly not explain his understanding in these terms. He seems to be explaining only a particular case. But his explanation draws upon, or perhaps, better, is assembled from a more general sort of knowledge. There are inseparable aspects of concrete and abstract thinking going on. Even if Pedro does not yet understand all the conventions involved in the decimal number notation, he now grasps the relationships among the mathematical objects involved in the two notational systems. He achieves this in part through his own efforts but also due to the collaboration and scaffolding provided by his colleague and the interviewers.

This example helps us in the analysis of the links between everyday and school mathematical knowledge. As happens in everyday life, the situation was meaningful and challenging for the children, triggering, in Hatano and Inagaki's (1992) terms, motivation for comprehension. The interviewers and each child's questions, comments, and suggestions as well as the goals of the activity provided the social setting for reflection and discussion. Children profited from their previous understand of division with remainder, an understanding that is compatible with the characteristics of everyday multiplication and division where whole numbers predominate and values smaller than one are avoided. The questions posed by the interviewers and the use of a calculator to help their computations, however, required them to go beyond their everyday experience and their previous understandings in order to relate what they knew about division with a remainder to the conventions for representing decimal numbers.

As the above dialogue seems to show, numbers without explicit reference to physical objects or their associated quantities can nonetheless be meaningful. It is not clear whether numbers ever become totally unlinked from the concrete circumstances associated with their emergence as mathematical objects. It is possible that even sophisticated mathematicians maintain certain ties to physical quantities and actions upon physical quantities. However, I think it is likely to be the case that objects formerly intimately attached to specific situations can gradually become functionally autonomous. That is to say, a student should be able to reflect upon and manipulate the newly emerged objects without having to constantly refer back to the circumstances in which they were initially introduced. Schools would appear to play a decisive role in this process by gradually removing supports initially essential for the representation of certain relations and also by providing notation and other external representations that allow the new conceptual objects to be discussed and otherwise acted on in ways clear to oneself and others.

Bringing everyday resources into new understandings: Graphs, Zefinha, and the presidential elections

But recognizing the importance of schools does not diminish the importance of the prior knowledge students bring into mathematics instruction from their vast and rich experience from daily life outside the classroom. We may have underestimated, in past studies of everyday mathematics, the fundamental role of schools. But this does not mean that people have fewer resources than we supposed. In fact, I believe that everyday mathematics constitutes an even far broader and deeper source of knowledge and intuition than we had originally thought. Let us consider a final example to explore these points.

Throughout discussions and analysis of data with David Carraher, Steve Monk, Ricardo Nemirovsky, Tracy Noble, Cornelia Tierney, and Tracey Wright, we came to the conclusion that, although most everyday mathematics studies have focused on arithmetic, there are reasons to suppose everyday situations, even though not including direct experiences with graphs, provide people with resources that could be relevant to the understanding of graphs.

Graphs are conventional symbolic systems often misunderstood even by advanced students. Traditionally teaching how to construct and interpret graphs is conceived in terms of methods to represent quantities according to certain rules. Recent research on how children and adolescents come to understand graphical information show, however, that people bring into the task of understanding information in graphs a wealth of previous experiences, knowledge, and intuitions that allows them to construct meaning for graphical representation without receiving specific instruction on how to plot points in a two-dimensional space (see Monk & Nemirovsky, in press; Nemirovsky, 1994; Tierney, Nemirovsky, Wright, & Ackerman, 1993).

The opportunity to discuss graphs with people with restricted school background happened on Aug. 20, 1994, when the Jornal do Brasil published the results of several months of opinion polls concerning Brazilian presidential elections to be held in October of that year. The results were displayed in line-graph format, with one line corresponding to each of the candidates. The x-axis displayed the dates of the polls; the y-axis displayed the percentage of votes going to candidates. A drawing of each candidate's face appeared to the right side of his graph line and, along each line, the percentage of vote intentions attributed to each candidate was displayed at each point. The two candidates leading the polls were Lula and Fernando Henrique, with the other candidates far behind. For the seven polls from March to August (with two of them in July), Lula's percentages were 28, 35, 41, 36, 34, 30, and 27. Fernando Henrique's corresponding percentages were 7, 16, 17, 17, 20, 29, and 40. The consistent improvement of Fernando Henrique's performance over the months culminates in a switch that was still unknown to many: Lula is no more the favorite candidate as was the case for the last five months, but Fernando Henrique is ahead of Lula (40% vs. 27%).

D. Carraher, Schliemann, and Nemirovsky (in press) describe one of the interviews I conducted and show how, with very limited school experience and no training on graphical representation, one can make sense of graphical information. The subject interviewed is Zefinha, a member of the cleaning staff of a Brazilian University, who had attended school for no more than three years and did not know that Lula, her candidate, was no more leading the polls.

I first explained to Zefinha that the lines in the graph were telling us who the people were saying they would vote for and that each line represented the votes each candidate was getting. Let's see how Zefinha reacted when I asked her to tell me what she thought the lines in the graph were saying:

> Interviewer: ...What do you think these lines are saying?
> Zefinha (without looking at the graph): They're saying that Lula's going to win.
> I: We're rooting for him, right?
> Z: Yeah, me too.
> I: Now tell me why do you think the line says he's going to win.
> Z. (again, not paying attention to the graph): Because there's no doubt that he'll win.

As one can see, her initial interpretation does not take into account the information on the graph but is rather the expression of her belief and wish that Lula would win. When I asked her to point out in the graph what showed that Lula has more votes she does some visual search along the lines in the graph and points to the endpoints (27 and 28) of Lula's line. In doing so, she seems to be grasping the general visual features of the graph and stressing, as was the case with another pilot subject, the idea that the whole trajectory counts. I then asked her to make systematic comparisons between the results for Lula and for Fernando Henrique for each poll date. Zefinha easily makes these comparisons:

> I: When Lula had 28, where was Fernando Henrique?
> Z: Fernando Henrique had 7.
> I: Lula was way out front, right?
> Z: That's what I want, for him to move ahead.
> I: Here (in May's column) Lula has how much?
> Z: Lula has 41.
> E: And Fernando Henrique?
> Z: 17.
> I: (Pointing to the x-axis first date) This was in March. Do you see? In the month of March, Lula was in front (em frente) and Fernando Henrique was below (em baixo).
> I: And afterwards? Lula, here (in May), for example?
> Z: 41
> I: He went up, huh? Now Fernando Henrique was here.

Z: 17.

As we approached the most recent polls Zefinha must take into account data that are against her wishes:

> I: What happened here (pointing to the number 34 in the first July poll)? Lula had here (in May) 41. What happened by the time he got here?
> Z: He went to 34.
> I: He lost a little?
> Z: Yeah.
> I: But is he still in front?
> Z: But he's still in front; he still has 28 (Lula's ratings in March poll) and this one 27 (August poll).
> I: 27.
> Z: And this 27 is Lula's....And he (Fernando Henrique) has 40.

Consistent with her previous indication that the whole line shows how well a candidate is doing, she seems to interpret the points in the lines as belonging to a candidate: Lula has 28 from March, "and this 27 is Lula's... and he (Fernando Henrique) has 40". This interpretation is not an absurd one and would be correct if we were, for instance, discussing the number of goals accumulated by the Brazilian team along the different games in the last world soccer championship. But for the election polls the only results that counts are the final results. This is what Zefinha will have to realize next as she tries to focus on August's results, the time when Lula is no more leading the polls:

> I: (Pointing to the last results) This is in August.
> Z: In August.
> I: Now in August, who's in front? Lula or Fernando Henrique? ...
> Z: In August? (Zefinha explores the graph and turns to the bottom line) Fernando Henrique is *here*! (points to a lighter blue line at the bottom of the graph).

Once more Zefinha tries to find features in the graph that would support here view that Lula is winning. From the comparisons along the different months she now seems to grasp that the last results are the ones that really matter. She also knows that lower lines indicate poorer performances. As a last resource to support her wish that Lula is winning she looks for Fernando Henrique's data there. The interviewer explains that the bottom lines refer to the other candidates and that what matters are the two top lines. Zefinha finally realizes that her candidate is behind:

> Z: It's Lula. Oh no! (with exasperation). It's Fernando Henrique?! No, but I don't want Fernando Henrique to win!
> I: (Empathetically) But it'll change back again.
> Z: (Now paying close attention to the fact that Fernando Henrique's line ends above Lula's) Because by this alignment it's Fernando Henrique that's above ("em cima").

Note that, although upset about the situation, Zefinha accepts that this is the information depicted in the graph. I then asked her to explain what should happen in September for Lula to win. She first answers that everyone should go out and work for more votes for Lula and then attempts to explain what she would like to see happening in the graph:

> Z: This little line has to give more votes, it has to go up more numbers.
> I: It's like this. So where do we want it to go? Show me.
> Z: We want it to go up like that (tracing the end segment of Fernando Henrique's graph).
> I: It has to go up. And what do you want to happen to Fernando Henrique's line?
> Z: I want Fernando Henrique's line to stay underneath his (Lula's).
> (From D. Carraher, Schliemann, & Nemirovsky, in press).

This is a good example of what Ricardo Nemirovsky calls "fusion" in understanding graphs. When Zefinha states that "This little line has to give more votes, it has to go up more numbers", she is fusing certain properties of the graph with the properties of the events she would like to see taking place: the graph serves as a metaphor for expressing trends and relations regarding voting preferences and voting trends can be expressed through the spatio-geometric properties of the graph. The interview also provides hints about how mathematical knowledge can be developed through reflection about the characteristics of new systems of representation by bringing into play previous knowledge, experiences, and beliefs, even when they do not directly correspond to the specific features of the case under discussion. Without previous formal instruction on the conventions of graphical representation, during the course of a 10 minute interview, Zefinha worked out many issues concerning how events can be represented through the visual and numerical characteristics of the graph. In this process she constantly brought into play her wishes and beliefs about the political campaign, as well as previous everyday knowledge. But in this case we are not talking about everyday knowledge acquired through specific experience with graphs, but rather about everyday knowledge that is more general and refers to an individual experiences and understandings about spatial relations, time flow, language, number properties, etc.

Implications for teaching: What to do in schools?

The contribution of school instruction to the development of more complex approaches to mathematical concepts and representations is an important one for many reasons. Schools can provide a much wider range of situations and tools for use of mathematical concepts and relations, allowing for children to explicitly focus on these from different perspectives. Schools also provide access to a variety of symbolic representations such as written symbols, diagrams, graphs, and explanations, which constitute opportunities for children to establish explicit links between situations and representations that would otherwise remain unrelated. It is in schools that children can come to understand mathematical concepts as belonging to, in Vergnaud's (1990) terms, conceptual fields.

The general recommendation that is frequently put forward when the relevance of everyday mathematics to school mathematics is considered has been that we should start from what children already know. But how would we proceed to take them to develop new strategies and new understandings? How can we benefit from what children already know avoiding, at the same time, limitations that are typical of the everyday cultural situations where cognition takes place? To bring to the classroom problems that can be related to their everyday practice does not seem to be the answer since these will also be limited and will not help exploring new facets of mathematical knowledge which are not part of everyday situations. Moreover, once transposed to the classroom cultural setting the problem is no more the same.

The available research data suggest that, for meaningful learning to take place in the classroom, reflection upon mathematical relations must be embedded in meaningful socially relevant situations where mathematics is used as tools to achieve relevant goals. But meaningful situations are not and must not be restricted to those that take place in out of school environments. School meaningful situations must allow for a wider variety of concepts and discovery of features that are not usually involved in street situations. We need school situations that are as challenging and relevant for school children as getting the correct amount of change is for the street seller and his customers. And such situations may be very different from everyday situations.

We provided examples of how children or adults can be challenged to use and understand mathematical principles and properties in situations that are very far from the everyday situations they cope with. The problems to be dealt with by Taís and Pedro were rather

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abstract since, differently from everyday situations, numbers were being used without specific referents. The situation, however, allows for discussion and discovery of mathematical properties as they are used as tools to break the code in a game, a goal that was genuinely challenging for the students participating in the activity. In the second example we showed that one can come to understand new systems of representation bringing previous knowledge and beliefs to situations that are not related to specific everyday work experiences. We showed how, without specific formal instruction related to the conventions used in graphs one can understand the information depicted in a graph when the facts under discussion are relevant enough to involve one's sense making resources in the search for understanding.

It is by bringing previous "adequate" or "inadequate" knowledge into the process of sense making of new situations that people come to develop more advanced mathematical understandings. Therefore, the answer to what to do in mathematical classrooms is not to design teaching activities that are copies of everyday activities, even if, in the classroom, one aims at profiting from the positive aspects of everyday understanding and use of mathematical knowledge avoiding, at the same time, the limitations imposed by the specific characteristics of everyday situations. It seems rather that both the "adequate" and "inadequate" conceptions that people bring from their everyday experiences are important steps towards the development of new mathematical concepts. Classroom activities should therefore aim at engaging students in using all of their resources in order to understand new situations. And these new situations, most often, are not the ones to be found outside of schools.

Can we dream of schools where students are permanently engaged in bringing into play all of their cognitive resources to understand new procedures, new symbolic systems, new mathematical relations? It is certainly not an easy task. But together, teachers and researchers in mathematics education are getting closer and closer to make the dream come true.

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Cognitive Growth in Elementary and Advanced Mathematical Thinking

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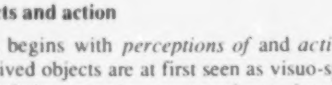
This paper addresses the development of mathematical thinking from elementary beginnings in young children to university undergraduate mathematics and on to mathematical research. It hypothesises that mathematical growth starts from **perceptions of**, *and* **actions on**, *objects in the environment. Successful "perceptions of" objects lead through a Van Hiele development in* **visuo-spatial** *representations with increasing verbal support to visually inspired verbal proof in geometry. Successful "actions on" objects use* **symbolic** *representations flexibly as "***procepts***" —* **pro***cesses to do and con***cepts** *to think about — in arithmetic and algebra. The resulting cognitive structure in elementary mathematical thinking becomes advanced mathematical thinking when the* **concept images** *in the cognitive structure are reformulated as* **concept definitions** *and used to construct* **formal concepts** *that are part of a systematic body of shared mathematical knowledge. The analysis will be used to highlight the changing status of mathematical concepts and mathematical proof, the difficulties occurring in the transition to advanced mathematical thinking and the difference between teaching and learning the full process of advanced mathematical thinking as opposed to the systematic product of mathematical thought.*

Perception, thought and action

I find it useful to separate out three components of human activity as input (perception), internal activity (thought) and output (action):



This simple observation allows us to see mathematical activities as perceiving objects, thinking about them, and performing actions upon them. I shall begin by considering input and output before moving on to the nature of the internal processing.

Input and output – objects and action

Elementary mathematics begins with *perceptions of* and *actions on* objects in the external world. The perceived objects are at first seen as visuo-spatial gestalts, but then, as they are analysed and their properties are teased out, they are described verbally, leading in turn to classification (first into collections, then into hierarchies) the beginnings of verbal deduction relating to the properties and the development of systematic verbal proof (Van Hiele, 1959).

On the other hand, *actions* on objects, such as counting, lead to a different kind of development. Here the *process* of counting is developed using number words and symbols which become conceptualised as number *concepts*. This leads to fundamentally different kind of development, described by Piaget, as follows:

> ... mathematical entities move from one level to another; an operation on such "entities" becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by "stronger" structures.
>
> (Piaget, 1972, p. 70)

Such an idea has led to a number of theories which highlight the duality of process and concept. Davis (1975) noted children may not distinguish between the name of a symbol and the underlying process. Skemp (1979) proposed a general "varifocal theory" in which a schema seen as a whole is a concept and a concept seen in detail is a schema. Greeno (1983) focused on the notion of "conceptual entities" which may be used as inputs to other procedures. More recently, Dubinsky (1991) speaks of ***encapsulation*** of process as object, Sfard (1991) of *reification* of process as object, and Gray & Tall (1994) see the symbol as pivot between ***pro***cess and con***cept***—the notion of ***procept***.

The two sequences of development beginning with object and action are quite distinct. I therefore hypothesise that, rather than view growth in elementary mathematics as a single development in the manner of a neo-Piagetian stage theory, an alternative theory is to see two different developments which occur at the same time. One is visuo-spatial becoming verbal and leading to proof, the other uses symbols both as processes to do things (such as counting, addition, multiplication) and also concepts to think about (such as number, sum, product).

It is interesting to note that these developments can occur quite independently. The Ancient Greeks developed a theory of geometry (including geometric constructions of arithmetic) without any symbolism for algebra and arithmetic, and it is possible to develop arithmetic and algebra without any reference to geometry. However, many useful links have been made between visual and manipulative symbolic methods and it is clearly opportune to take advantage of them to develop a versatile approach which uses each to its best advantage.

In the advanced stages of such a development, certain subtle difficulties occur which mean that advanced mathematical thinking must expunge itself of possible hidden assumptions that occur when visual ideas are verbalised. In the nineteenth century a number of flaws became apparent in Euclidean geometry and theoretical developments in algebra (such as non-commutative quaternions) were over-stretching simple beliefs in the manipulation of symbols. Research mathematics took a new direction using set-theoretic definition and logical deduction. Theorems inspired by geometric perception and symbolic manipulation were reformulated to give a new axiomatic approach to mathematics that led on to greater generality.

This theory is also flawed. The axiomatic method asks us to write down finite lists of set-theoretic definitions and axioms and to deduce theorems in a finite number of steps. But if we do this with an infinite set, such as the natural numbers, Gödel showed that there are theorems that must be true but which cannot be proven in a finite number of steps. Essentially, there will always be "too many theorems" to prove. Thus the existence of a systematic body of formal mathematical knowledge is not the final quest in mathematics, although it does offer a vital foundation upon which even more sophisticated ideas can be built.

Advanced mathematical thinking today involves using cognitive structures produced by a wide range of mathematical activities to construct new ideas that build on and extend an ever-growing system of established theorems.

The cognitive growth from elementary to advanced mathematical thinking in the individual may therefore be hypothesised to start from "perception of" and "action on" objects in the external world, building through two parallel developments—one visuo-spatial to verbal-deductive, the other successive process-to-concept encapsulations using manipulable symbols—leading to a use of all of this to inspire creative thinking based on formally defined objects and systematic proof (figure 1).

Internal processing and external representations

The cognitive growth that occurs in mathematics is implicitly designed to make maximum use of the facilities available to *homo sapiens*. The two parallel developments described relate to the complementary roles of perception (input) and action (output). In between is the internal mental processing which is far more difficult to describe and analyse. Crick suggests that:

> The basic idea is that early processing is largely parallel – a lot of different activities proceed simultaneously. Then there appear to be one or more stages where there is a bottleneck in information processing. Only one (or a few) "object(s)" can be dealt with at a time. This is done by temporarily filtering out the information coming from the unattended objects. The attentional system then moves fairly rapidly to the next object, and so on, so that attention is largely serial (i.e., attending to one object after another) not highly parallel (as it would be if the system attended to many things at once). (Crick, 1994, p. 61)

Brain activity therefore has two highly contrasting features:

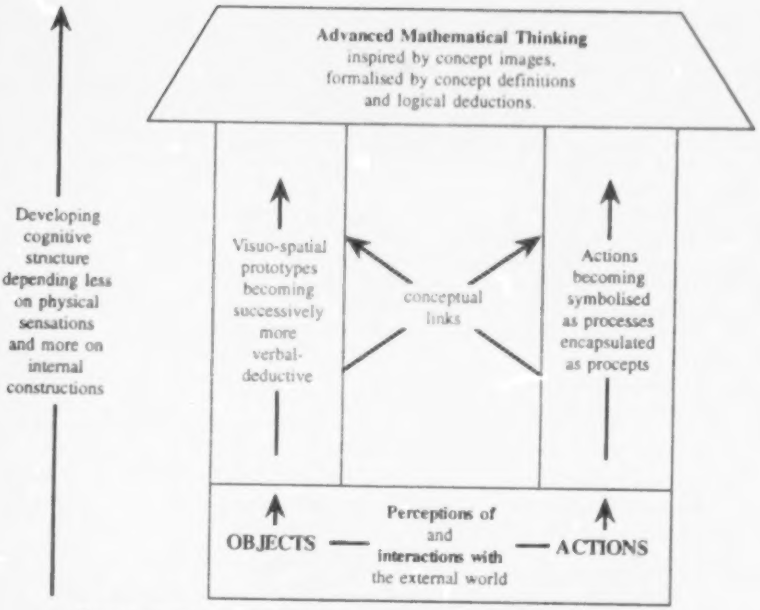
- a huge store of experiences and simultaneous activity,
- a small focus of attention,

(where the latter need not be a *place* in the brain to store items as in a computer but a mental activity which is temporarily linked to conscious thought processes).

To minimise the cognitive strain it is essential to do two things:

- compress knowledge appropriately for the small focus of attention,
- construct linkages to other mental data to make it easy to use.

The first is an essential characteristic of mathematics:



Advanced Mathematical Thinking
inspired by concept images,
formalised by concept definitions
and logical deductions.
Developing cognitive structure depending less on physical sensations and more on internal constructions
Visuo-spatial prototypes becoming successively more verbal-deductive
conceptual links
Actions becoming symbolised as processes encapsulated as procepts
OBJECTS
Perceptions of and interactions with the external world
ACTIONS

Figure 1: Outline cognitive development from child to research mathematician

> Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics. (Thurston, 1990, p. 847)

It is achieved in a variety of ways—routinising processes so that they occupy little conscious attention, using pictures to allow the viewer to focus at whatever level and on whatever detail is desired, and using words and symbols (particularly procepts) to compress the notation into small, mentally manipulable entities.

The second involves the development of *conceptual knowledge* with many links to maximise retrieval. This also involves concept-process links enabling the successful individual to carry out mathematical procedures to find answers to problems. However, if the mathematics places too great a cognitive strain, either through failure to compress or failure to make appropriate links, the fall-back position resorts to the more primitive method of routinising sequences of activities —rote-learning of *procedural knowledge*.

The status of mental objects

In cognitive growth, the mental objects we think about are constructed in several different ways, each having a different status. The visual objects we see are direct perceptions of the outside world, or rather, our own personal constructions of what we think we see in the outside world. Later in geometry, objects such as a "point" or a "line" take on a more abstract meaning. A point is no longer a pencil mark with finite size (so that a child may imagine a finite number of points on a line segment (e.g. Tall, 1980)), but an abstract concept that has "position but no size". A straight line is no longer a physical mark made using pencil and ruler, but an imagined, perfectly straight line, with no thickness which can be continued as far as required in either direction. In Euclid, a line is defined as "breadthless length" and a straight line "lies evenly with the points on itself". These words do not *define* a straight line in any absolute sense, but they help to convey the meaning of the perfect Platonic object which we may "see" lying behind any inadequate physical picture. As Hardy observed:

> Let us suppose that I am giving a lecture on some system of geometry, such as the ordinary Euclidean geometry, and that I draw figures on the blackboard to stimulate the imagination of my audience, rough drawings of straight lines or circles or ellipses. It is plain, first, that the truth of the theorems which I prove is in no way affected by the quality of my drawings. Their function is merely to bring home my meaning to my hearers, and, if I can do that, there would be no gain in having them redrawn by the most skilful draughtsman. They are pedagogical illustrations, not part of the real subject-matter of the lecture.
>
> (Hardy, 1940/1967, p. 125.)

The mental "objects" constructed by process-object encapsulation have a very different status. As Dörfler suggested:

> … my subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5•5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object "5".
>
> (Dörfler, 1993, pp. 146–147.)

In terms of the notion of *concept image* of Vinner (Vinner & Hershkowitz, 1980), there is no conflict here. Within our mental structure we have both recognition structures that recognise, say, the perceptions of a physical object, such as a drawing of a triangle, and we also have connected sequences of mental actions that are triggered to carry out processes in time. The concept image of a *procept* uses the symbol to links to suitable processes and relationships in the cognitive structure. Thus, although we may not have anything in our mind which is like a physical object, we have symbols that we can manipulate *as if* they were mental objects.

I do not believe in my own case that I have things in my mind that correspond to visualisations either. Despite working for many years on visualisations in mathematics in which I can produce good external pictures on the computer screen to represent mathematical concepts, the pictures I conjure up in my mind are very different from the external representations. It is different with words. As I type this, I can hear the words in my mind and if I start saying them as I type, what I hear out loud is what I hear in my head. But *homo sapiens* has no "picture-projecting facility" for communication in the same way as it has a verbal "sound-making facility".

I have a theory therefore that when we visualise, we use not "picture-making" facilities, but "picture-*recognising* facilities" which we have in plenty. We have many structures that resonate with incoming visual stimuli to recognise them and we simply use these recognition's to attempt to build up our visuo-spatial imagery. The result is a vague "sense" of a picture. Certainly in my case it is vague. I do not know what *you* see when you think of a visualisation, perhaps you see an eidetic image in full colour. Then again, what we all see may be just the emperor's new clothes!

Many mathematicians say that they think in "vague" visuo-spatial ways as a springboard for more abstract thinking. Hadamard (1945) reported that most of the mathematicians he consulted did so. In his own case he even saw formulae in this way:

> I see not the formula itself, but the place it would take if written: a kind of ribbon, which is thicker or darker at the place corresponding to the possibly important terms; or (at other moments), I see something like a formula, but by no means a legible one, as I should see it (being strongly long-sighted) if I had no eye-glasses on, with letters seeming rather more apparent (though still *not legible*) at the place which is supposed to be the important one.
>
> (Hadamard, 1945, p. 78.)

Representations

In considering the kind of mental "objects" we have in different mathematical contexts, it is interesting to return to the ideas of Bruner (1966) who formulated his theory of three different types of representation of human knowledge:

- enactive,
- iconic,
- symbolic.

One of these is essentially a physical *process* (enactive) whilst the other two produce physical objects that are drawn or written[1] (iconic, symbolic). Iconic representations drawn by hand, such as a free-hand graph, also have enactive elements in them, suggesting a broader "visuo-spatial" concept. (For instance, one senses enactively that a "continuous" graph going from negative to positive must pass through zero.)

Symbols as procepts in arithmetic, algebra etc., also have dual process-object meanings. This in turn suggests that the symbolic mode of presentation needs subdividing as Bruner himself hinted when he mentioned, "language in its natural form" and the two "artificial languages of number and logic", (Bruner, 1966, pp. 18, 19).

Natural language occurs throughout mathematics to set the mathematical activity in context. In the visuo-spatial to verbal development, natural language becomes a vehicle for describing iconic images and formulating proof. It can also be used to describe properties of numbers, for instance, that addition is commutative because it is observed to be always independent of the order (a fact easier seen by visualising the change in order than carrying out the counting procedure—a valuable use of the interrelations between visual and symbolic.) Meanwhile the "artificial language of number" has mental objects which are procepts and the "artificial language of logic" in advanced mathematics has concepts which are formally *defined*.

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[1] Verbal symbolism can, of course, also be spoken, but the written word has great value as a permanent record that can be scanned and reflected upon.

It is essential to distinguish between elementary mathematics, (including geometry) where objects are *described* and advanced mathematics where objects are *defined*. In both cases language is used to formulate the properties of objects, but in elementary mathematics the description is constructed from experience of the object, in advanced mathematics, the properties of the object are constructed from the definition—a reversal which causes great difficulties of accommodation for novices in advanced mathematical thinking.

This gives a range of different types of representation in mathematics, including:

- enactive (physical process),
- iconic (visual),

and three forms of symbolic representation:

- verbal (description),
- formal (definition),
- proceptual (process-object duality).

The notion of "procept" helps in the analysis of cognitive difficulties related to symbolism. When Eddie Gray and I first coined the term I felt, in a moment of self-doubt, that all we had done was to give a name to something that was well-known to the mathematics education community. Subsequently I realised it was more. By giving it a name, we had essentially *encapsulated the process of encapsulation*. This enables us to discuss different kinds of encapsulation in different contexts and to see how learners face cognitive difficulties when procepts behave differently in different contexts.

For instance, in the development from the process of counting to the number concept, the sequence of number words initially only function as utterances in the schema of pointing and counting, but then the last word becomes the name for the number of objects in the collection. In arithmetic of whole numbers, symbols such as 4+3 initially evoke a counting procedure (count-all) which is then compressed via "count-on" (which uses 4 as a number concept and +3 as a count-on procedure) to a "known" fact where 4+3 is the number 7. In this encapsulation there is a new *concept*, namely the *sum*, 4+3, but it relates to a known *object* (the number "7"). However, for the process of equal sharing for 3/4 (divide into four equal parts and take three) to be encapsulated requires the construction of a *new* mental object — a fraction. Hence the considerable increased difficulty with fractions as a succession of encapsulations and mental constructions.

Arithmetic procepts such as 4+3, 3×4, have a built-in algorithm to compute the result, which children come to expect. Such procepts are genuinely *operational*, in the sense that one can operate on them to get an answer. But in algebra, procepts such as $4+3x$ certainly have a process of evaluation (add four to three times x) but cannot be evaluated until x is known. Such symbols are termed *template* procepts, in that they are templates for operations which can be evaluated only when the variables are given appropriate values. However, the symbols can still be manipulated as objects, in simplifying, factorising, solving equations, and so on. The shift in focus from the symbolism of arithmetic where the aim is to obtain numerical answers to the manipulation of template procepts in algebra is one which causes severe difficulties for many learners.

Likewise, in the beginnings of calculus there are symbols which act dually as process and concept. For instance, the *limit* procept:

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

dually represents both *process* (as h gets small) and *concept* (the limit itself). This causes further difficulties because it is not computed by a finite set of calculations (Cornu, 1991). Instead, in a specific case such as $f(x)=x^2$, first the simplification is performed assuming $h\neq 0$, then the final result is computed by setting $h=0$. For other expressions this confusing limit process soon becomes too complicated and derivatives are computed by a collection of rules.

The limit concept causes great difficulties for students (Cornu, 1991, Williams, 1991). The majority seem to continue to treat a limit as a *process* getting close rather than a *concept* of limit. The usual default behaviour to cope with lack of meaning is to use the rules of differentiation procedurally. It at least has the familiar quality that it is an algorithm giving a result, albeit a symbolic one, making the limit procept operational.

Figure 2 uses this analysis of different forms of representation to show how they feature in different mathematical topics. It outlines the visuo-spatial to verbal development in geometry, the proceptual development in arithmetic and algebra, and the relationships between them in measurement, trigonometry and cartesian coordinates.

At the top of the figure are the subjects which begin the transition to advanced mathematical thinking. All of these require significant cognitive reconstructions. Euclidean proof requires the realisation of the need of systematic organisation, and agreed ways of verbal deduction for visually inspired proof (the use of congruent triangles). The move into calculus has the difficulties caused by the limit procept. The move into more advanced algebra (such as vectors in three and higher dimensions) involves such things as the vector product which violates the commutative law of multiplication, or the idea of four or more dimensions, which overstretches and even severs the visual link between equations and imaginable geometry.

The transition in all three subjects therefore requires considerable cognitive reconstruction involving a struggle to understand. However, there is an even greater leap to be made in advanced mathematical thinking to formal definitions (which changes the status of the objects being studied) and formal deduction (which changes the nature of proof). To see just how much change is required, let us briefly look at how the nature of proof is dependent on the representations available and on the mathematical context.

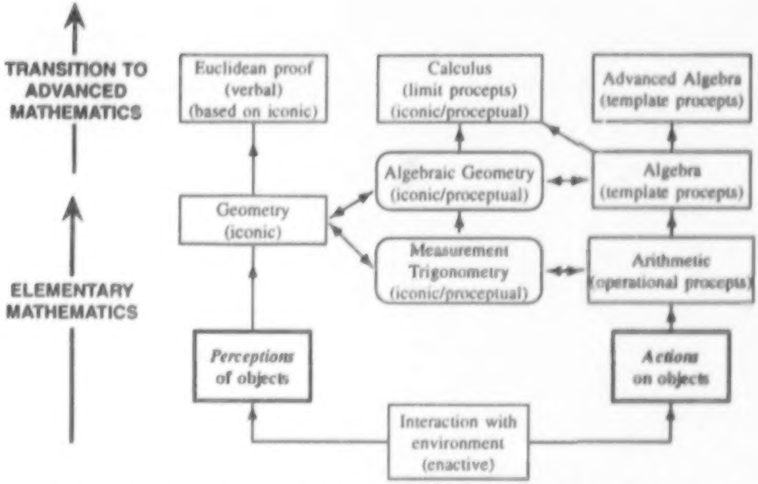

TRANSITION TO ADVANCED MATHEMATICS
ELEMENTARY MATHEMATICS
Euclidean proof (verbal) (based on iconic)
Calculus (limit procepts) (iconic/proceptual)
Advanced Algebra (template procepts)
Algebraic Geometry (iconic/proceptual)
Algebra (template procepts)
Geometry (iconic)
Measurement Trigonometry (iconic/proceptual)
Arithmetic (operational procepts)
Perceptions of objects
Actions on objects
Interaction with environment (enactive)

Figure 2: Actions and objects in the building of various mathematical knowledge structures

The Status of Proof

Given different types of representation and different ways of thinking about them, it follows that there are likely to be different kinds of proof. In the *enactive* mode, proof is by prediction and physical experiment: to show two triangles with equal sides have equal angles, put them one on top of another and see. In the *iconic* mode, a picture is often seen as a *prototype*, that can be thought of as representing not only a single specific case, but others in the same class. The picture in figure 3, which demonstrates that four times three is three times four will work for any other whole numbers and so may be visualised as a *generic* proof that whole number multiplication does not depend on the order:

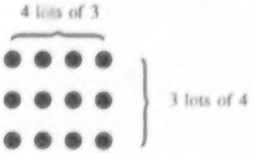

4 lots of 3
3 lots of 4

Figure 3: Multiplication is independent of order

Visual proofs, however, begin to fail when pictorial prototypes cease to represent the full meaning of the class of objects to which the proof refers. For instance, the difference between real numbers and rational numbers is difficult to represent visually (although I simulate it in some of my own software for schools, (Tall, 1991)). Here are two pictures. The one on the left is of a *continuous* function on the rationals (the formula reads "if $x^2>2$, then the value is 1 else it is -1, on the domain where x is rational). The one on the right is the real function taking the value $x^2(x^2-1)+1$ if x is rational, and 1 if x is irrational. It is continuous only at $x=-1$, 0 and 1. (It is even differentiable at $x=0$.)

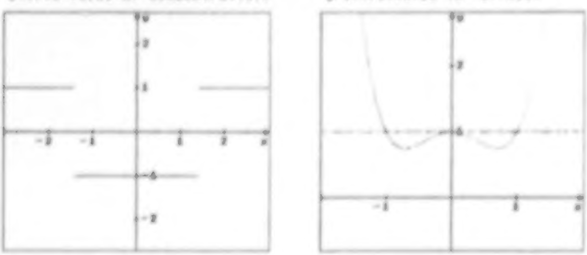


Figure 4: The first function is continuous, the second continuous at −1, 0, 1.

Developing meaning for pictures so that they give correct intuitions is a sophisticated business, which is even more difficult to turn into formal proof. However, for some professional mathematicians visualisation gives valuable insight in the sense of Dreyfus (1991), whilst others, aware of the possible pitfalls, distrust them completely.

Verbal proof depends on the context in which it occurs. For instance, in Euclidean geometry it is essentially a translation of visual generic proofs for triangles and circles based on the pivotal notion of congruent triangles. It is *not* logical proof in the sense accepted in modern axiomatic mathematics. However, it does introduce the learner to a most important aspect of axiomatic proof, that of *systematic organisation*, proving theorems in order, so that each depends only on previously established theorems.

Proof involving *procepts* is usually performed through using the built-in processes. For instance, proof in arithmetic is either through a generic computation, "typical" of a class of examples, or using algebraic computation, e.g. the proof that the sum of two successive odd numbers ($2n+1$ and $2n+3$) is a multiple of 4 ($4n+4$).

Proof at the *formal* level consists essentially of rearranging the content of a given set of quantified statements to give another quantified statement. These statements relate to definitions of formal mathematical concepts, so if certain properties of concepts are given, others are deduced. The logical part of the deduction is just the tip of the iceberg. The part under the water is a hazard for many trying to navigate for the first time. Expert mathematical thinkers use so much more of their experience to choose concepts worth studying, to formulate them in the most productive way and to select likely lines of attack for proof.

> A group of mathematicians interacting with each other can keep a collection of mathematical ideas alive for a period of years, even though the recorded version of their mathematical work differs from their actual thinking, having much greater emphasis on language, symbols, logic and formalism. But as new batches of mathematicians learn about the subject they tend to interpret what they read and hear more literally, so that the more easily recorded and communicated formalism and machinery tend to gradually take over from other modes of thinking. (Thurston, 1994, p. 167).

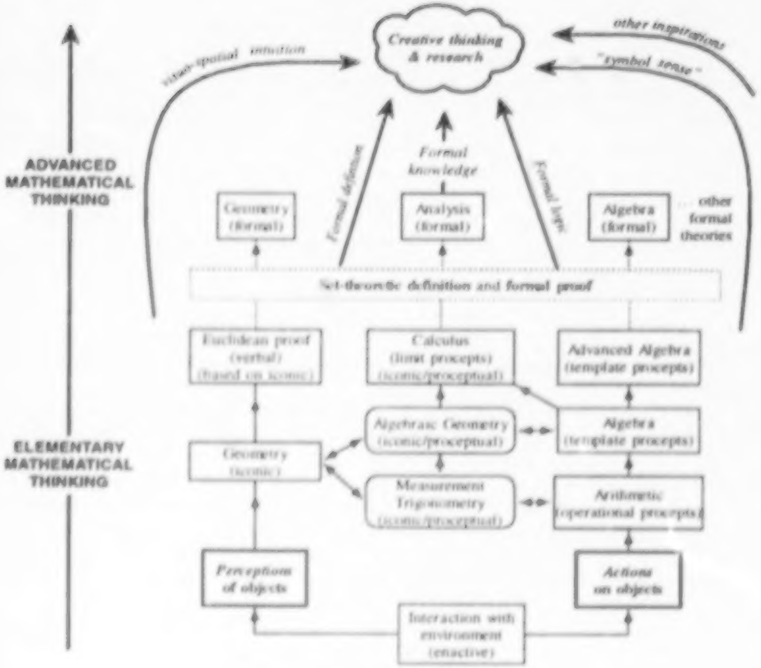
The move to advanced mathematical thinking, using a full range of personal mental imagery to develop new theories formulated in terms of systematic proof is more than just the appreciation of a formal development from definitions and axioms. It builds in the kind of structure exhibited in figure 5, with the advanced mathematical thinker using visuo-spatial ideas, symbol-sense and all kinds of intuitions to develop new theories that can be woven into the Bourbaki-like systematic development that forms the solid theoretical basis of the subject.

Where is the transition to advanced mathematical thinking?

In the description so far, the place where elementary mathematical thinking becomes advanced has yet to be precisely defined. In figure 1, the "transition to advanced mathematics" includes systematic Euclidean geometry, calculus and advanced algebra. Certainly these subjects all involve inherent difficulties requiring considerable cognitive reconstruction and, at various times in history (ancient Greece, the seventeenth and nineteenth centuries, respectively), they were topics of mathematical research by the most creative minds of their generation. Calculus and advanced algebra also contain a significant quantity of the mathematics taught at university for students as service subjects, so it would be politic to include these subjects as "advanced mathematics".

In the deliberations of the Advanced Mathematical Thinking Group of PME at its first meeting in 1987, we found it impossible to come to an agreement and decided pragmatically to take our brief to study mathematical thinking in topics beyond regular mathematics from the age of sixteen. Pragmatism suggests that it would be pertinent to include Euclidean geometry, calculus and advanced algebra above the line. However, whereas each of these subjects has its own idiosyncratic difficulties, the universal cognitive change occurs with the introduction of the axiomatic method, where mathematical objects have a new cognitive status as defined concepts constructed from verbal definitions. This is therefore a more natural place to draw the line between elementary and advanced mathematical thinking. It is essentially a change in cognitive stage from the equilibrium of visual conviction and proceptual manipulation to defined objects and formal deduction.

It remains valuable to consider the first level beyond elementary school mathematics to be a preliminary stage of advanced mathematical thinking, in which elementary ideas are stretched to their limits (literally!) before the theoretical crisis they generate requires the reconstruction of a formal view. Many will not require the full range of formal mathematics, being fully occupied with the proceptual complexities of the manipulation of symbols in calculus and algebra. The full range of creative advanced mathematical thinking is mainly the province of professional mathematicians and their students.



ADVANCED MATHEMATICAL THINKING
ELEMENTARY MATHEMATICAL THINKING
Creative thinking & research
visuo-spatial intuition
other inspirations
"symbol sense"
Formal definition
Formal knowledge
Formal logic
Geometry (formal)
Analysis (formal)
Algebra (formal)
... other formal theories
Set-theoretic definition and formal proof
Euclidean proof (verbal) (based on iconic)
Calculus (limit procepts) (iconic/proceptual)
Advanced Algebra (template procepts)
Algebraic Geometry (iconic/proceptual)
Algebra (template procepts)
Geometry (iconic)
Measurement Trigonometry (iconic/proceptual)
Arithmetic (operational procepts)
Perceptions of objects
Actions on objects
Interaction with environment (enactive)

Figure 5 : The development of advanced mathematical thinking

The relationship between elementary and advanced mathematical thinking

The changes in status of mathematical objects and mathematical proof at various stages of development offer an alternative viewpoint to consider the relationship between elementary and advanced mathematical thinking. Indeed, it reveals very different forms of mathematics in school and university. The "New Math" of the nineteen sixties was an attempt to introduce a set-theoretic deductive approach in elementary mathematics and it failed. Now mathematics educators involved with mathematics in school are operating in an age of democratic equality of opportunity which is predicated on a broad curriculum suitable for the needs of the wide population. There are signs that the curriculum in elementary mathematics is producing students less ready to study mathematics at university.

A recent report (Pozzi & Sutherland, 1995) has highlighted perceived shortcomings in students arriving in the UK to study engineering. An exchange of letters in the English national press has revealed serious concerns about "falling standards" related to changes in the English curriculum. For example, Sykes & Whittaker (1994) report that in 1994, only 50% of the entrants to their business studies course could multiply 1/2 by 2/3 and, whereas 66% could correctly calculate the square of 0.3 in 1987, by 1994 this had fallen to 16%. The general consensus amongst university mathematicians in England is that students arrive at university to study mathematics with less understanding of proof, less proficiency in handling arithmetic (particularly fractions and decimals) and less facility with algebraic manipulation.

The decline of Euclidean geometry in English schools has led to a loss of experience with systematic proof. The increase in practical links with real world problems and loss of manipulative practice seems to lead to less meaning *within* mathematics. Procepts, such as fractions, involve many conceptual encapsulations, including the encapsulation of counting as the concept of number, addition of whole numbers as sum, repeated addition as product and the process of equal sharing as the concept of fraction. There is little wonder that fractions proves difficult for a wide range of the population. Likewise, the meaningless manipulation of symbols in algebra is a consequence of inability to give them meaning as process and concept (Sfard & Linchevski, 1994).

It would be pertinent for a proportion of the mathematics education community to focus on the learning of those students in elementary mathematics who might develop the potential for advanced mathematical thinking, to analyse whether their learning environment is suitable for their long-term development. Short-term it would be possible to consider the ways in which the "more successful" do mathematics, to see if they need a different environment from others. Perhaps some of the educational devices for introducing mathematics in an elementary way, such as physical balances to introduce equations, introduce cognitive baggage which is not in the long-term helpful for cognitive compression.

Advanced mathematical thinking and undergraduate mathematics

At college level, mathematics is usually still taught in the "definition-theorem-proof-illustration" sequence with little opportunity for developing a full range of advanced mathematical thinking.

> The huge quantities of work covered by each course, in such a short space of time, make it extremely difficult to take it in and understand. … From personal experience I know that most courses do not have any lasting impression and are usually forgotten directly after the examination. This is surely not an ideal situation, where a maths student can learn and pass and do well, but not have an understanding of his or her subject.
>
> *Final Year Undergraduate Mathematics Student*

Rote-learning at university is even worse than procedural learning in school. At least procedures can be *used*, even if the range of application is narrow, but a rote-learnt proof that has no link to anything else has little value other than for passing examinations.

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Regrettably, students who are good at routine problems in advanced mathematics often fail when faced with something a little different (e.g. Selden, Mason & Selden, 1994).

Mathematicians seem to face a dilemma:

> … we should not expect students to (re-)invent what has taken centuries of corporate mathematical activity to achieve. Yet if we do not encourage them to participate in the generation of mathematical ideas as well as their routine reproduction, we cannot begin to show them the full range of advanced mathematical thinking. (Ervynck, 1991, p. 53)

Fortunately, it *is* possible to encourage students to *think* in a mathematical way at university level, as is shown by problem-solving approaches such as Mason *et al* (1982), Schoenfeld (1987), Rogers (1988), and the "proof debates" of the Grenoble school (Alibert, 1988). Following the problem-solving approach of Mason *et al*, Mohd Yusof (1995) has shown that such a problem-solving approach changed student attitudes in a way that university professors desired, whereas the adherence to traditional lecture methods and the vast quantity of rote-learnt content caused students to change attitude in the opposite direction. Typical responses from professors and students were as follows:

> I see mathematics as something that needs *doing* rather than learning where I should participate actively in making conjectures, constructing arguments to convince others, reflecting on my problem-solving and so on. But I think the maths course at the university does not encourage this.
> *Student A*

> We work under pressure and often feel anxious that we can't do maths. Not because we can't do it, because we can't do it in time.
> *Student B*

> The experience of making conjectures, generalising and the like I think students can get themselves on their own, from doing their project work. We do not have the time to teach them everything.
> *Professor C*

> To me mathematics is a mental activity, but I should say that at this level I present it more as a formal system. Because we are confined by the syllabus and also depending on the students' background. … I would like to change. How do I do that? I don't know.
> *Professor D*

Paradoxically, traditional ways of teaching are, for most students, causing precisely the opposite effect that university mathematicians desire. The sheer difficulty and volume of material to be covered in a university mathematics degree makes it difficult for students to cope with formal mathematical content in a limited time. But does this mean that we must accept the status quo of a huge formal syllabus with widespread rote learning, or is it not possible to modify courses to allow students to develop ways of thinking more mathematically? The acquisition of a wide repertoire of advanced mathematical thinking is a challenge which now faces university mathematicians. Is it a challenge which will be accepted?

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PANEL PRESENTATIONS

In the Panel, thirteen minutes of video footage will be used to anchor discussion around a common set of data.

The Panel is seen as an opportunity for participants to raise issues, to formulate diverse interpretations regarding the "same" happenings, and to construct their own arguments, however rigorous, however speculative, about the nature of mathematical thinking, based on their understanding of the understandings of two students and two interviewers.

Students exposed to the same physical stimuli (or the same tasks) in the classroom do not necessarily experience the same ideas and feelings, wonder about the same issues, derive the same conclusions, nor make the same associations. Likewise, researchers exposed to the same video segments do not necessarily see the same thing. The panelists (and the audience) bring their unique backgrounds to bear on the events depicted.

What follows are the actual materials made available to each of the discussants (including a written transcript of the dialogue from the video tape) and the papers written by Profs. Bartolini-Bussi, Meira, and Vergnaud.

[Special thanks to Barbara Fox and Tobin School of Cambridge, MA-USA, for making the video possible.]

TERC- Technical Education Research Centers
Cambridge, MA

This videotape comes from a 5th grade public school classroom in Cambridge, MA, where students have been working since mid-November on a 4-week pilot of the NSF-sponsored elementary math curriculum called: Investigations in Number, Data and Space, "Patterns of Change: Tables and Graphs". Students are more than halfway through the unit. During the first 2 Investigations students created "motion trips" on a meter tape by walking, running, stopping, etc. They represented these trips with motion stories, number tables and distance/time graphs. The students also spent a few days using a software program which allows them to invent motion trips that can be viewed along a linear path, number table, and/or a distance/time graph. One of the important goals of the unit is for students to be able to look at any of these representations of motion, describe the trip that took place and make another representation of the same trip.

In addition to piloting curriculum with this class, we are also doing research for the Mathematics of Change project. We videotape daily in class where students have math for about an hour a day. We also conduct individual and group clinical interviews, and analyze videotapes as well as student work. Another important aspect of the research is the weekly meeting with both teachers who are involved in the research and curriculum pilot.

A note on transcript conventions:

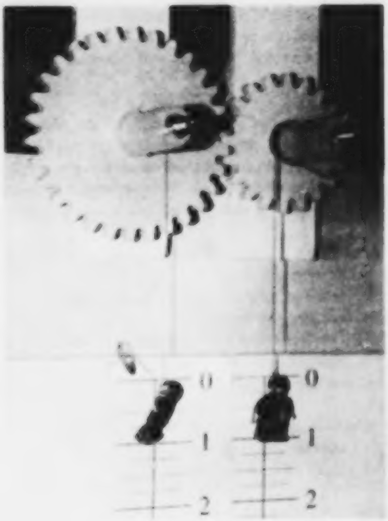
/	is used when someone is cut off or stops speaking abruptly
...	is used when someone's voice trails off
[ ]	is used to clarify a word (like it) or describe a gesture
( )	is used when someone has said something unintelligible
|	is used to indicate the beginning of an overlap when speakers are talking at the same time
,	a comma between 2 numbers is a way of separating them (3, 5 is three, five. Not three and five tenths)

Utterances are often placed on separate lines.

Introduction

Tony and Dennis (both age 10, grade 5) are about to undertake the task of creating a table of numbers about the position of 2 divers for each of 5 gear sets with different gear ratios. They will look for patterns in their tables and also graph some of these number tables in a position vs. position graph. We will follow them through this process for 1 set of gears (referred to as "A") by looking at 5 episodes of videotape which total about 13 minutes. The first 4 episodes are from the last 12 minutes of class on 12/15/94. The 5th episode is from the following day after class in an interview setting with Ricardo Nemirovsky. David Carraher, another interviewer, asks questions as he handles the camera.

This is the 3rd day that students have been working with gears (see picture below) and the first time they've worked with a gear set that is not a whole number ratio of 1:n. For example, they've seen 10:20, 20:40, 20:60, 10:60 (where the ratio is the ratio of the gear teeth and is 1:2, 1:2, 1:3, 1: 6), but this is the first time they've seen a 2/3 ratio (20 teeth to 30 teeth). Each gear wheel has an axle of the same diameter from which a small plastic person called a diver (sometimes called a driver by the children) is suspended on a string which is 12 units long. There is also a vertical scale that is 12 units long, marked with the numbers 0-12. Between each of the numbers are 3 lines (creating 4 sections) so that someone can measure, for instance, i6 and a half". Each unit is 4 centimeters long, which is half the circumference of the dowel which turns the gear wheel. Also available are yellow dot stickers (see photo) which can be used to mark places on the gear wheels.



Children have been told that they should measure from the top of the diver's head (although they sometimes invent their own system). They have also been advised to make sure that the 2 divers both start at 0. (In this case they have not actually started both divers at 0, and this is important for understanding the data table they produce.) The boys have also been shown how to take measurements of the divers' positions by moving the left diver down by increments of one unit. They have already collected the data above the dotted line in diagram 1(after writing 2 and 3.5) before Episode 1 begins:

DIAGRAM 1

L (Left Diver)	R (Right Diver)
0	0
1	2
2	3.5
3	5
4	6.5
5	8
6	9.5
7	11
8 1\4	12

Episode 1- Measuring and Recording data (00-3:20)

•TONY: Either way it's not going to be even.

•DENNIS: 5 [moving to number table]

•TONY: 5

•DENNIS: 3,5 [writing 3 and 5 on table]
So when he gets to 4, we'll see where he is.

•TONY: He [right diver] passed 4 already [winding divers up].

•DENNIS: No, when the other guy [left diver] gets to 4.
[Dennis presses both divers against scale.]
One more notch, one more, one more.[Tony winds divers down.]
6 and a half. [Dennis writes on number table.]

•TONY: 4,6

•DENNIS: 4, 6 and a half.

•TONY: 6 and a half.

•DENNIS: Oh yeah, I remember I did this yesterday.
What you do...When he's [Left Diver] at 5,
he's [Left Diver] not at 5 yet.

•TONY: Well, it's almost at... he's ( ) below 7.

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[Tony presses the right diver against the scale.]

•DENNIS: Wait, wait let go of it. [Dennis turns the gears and presses the left diver against the scale.]

•TONY: He's [Right Diver] almost at 8.

•DENNIS: Now he's [Right Diver] at 8.

•TONY: Yeah, so it's 8.
•DENNIS: 5,8.

•TONY: Yeah, 5,8. [Dennis writes on number table.]
When he's at...9.
[Tony presses right diver against scale with his marker.]

•DENNIS: [touching diver] 9 and a half.

•TONY: Yeah, 9 point 5.

•DENNIS: So, [looking] 6 and 9 and a half.

•TONY: 6 and 9 point 5.
[Dennis writes 6 and 9.5 and goes back to divers.]
Is that ten?

•DENNIS: No, keep going. One more.
OK. He's at 11.

•TONY: He's at 11? He's at 7. 11, 7.

•DENNIS: 7, 11. [writing numbers on table] 7, 11.
[Dennis goes back to gears.] OK, when. Keep.

•TONY: ...at 12. He's at 9.

•DENNIS: He's at...

•TONY: 8 point 4.

•DENNIS: [pressing R diver against scale] 8 and one fourth.

•TONY: Wait, why is there 2 blues? [markers]

•DENNIS: Jesse [another student] just put 'em there.
Wait there's one, two, three [lines on scale between numbers]. Yeah, 8 and one fourth. [Dennis writes 8 1\4 and 12]

•TONY: Yeah that's what I\ I was right...
Yeah, OK.
So we're pretty sure that one...
So which ones [other gear sets] do we have left [to do]?

•DC: Is there/ Is there a pattern here [number table A]?

•TONY: Uhh, yeah. Here? Which one?

•DC: What do you see?

•TONY: H, F or A? [Each letter refers to a different gear set.]

•DC: ( )

•TONY: Well, A, there/ He's [R] always like 1 or 2 ahead.
'Cause when he's [L] at 4, he's [R] at 6.5, so he's [R] 2.5 ahead of him.
When he's [L] at 5, he's [R] at, 2 ahead of him.
When he's [L] at 6, he's [R] um, 3.5 ahead of him.
When he's [L] at 7, he's um, 4 ahead of him [R] .
•DENNIS: 4
•TONY: When he's [L] at 8 and one fourth, he's [R] already finished.

In between Episode 1 and 2, Tony spontaneously tries to look for patterns in the tables they'd made from 2 gear sets with other ratios. David Carraher asks him to predict what some graphs of these number tables might look like. In Episode 2, Dennis and Tony are referring to the SAME set of gears (A) as in Episode 1.

(2 minutes later) *Episode 2 - Gears and Patterns (3:30- 4:40)*

•DC: Did those [gear set A, 20:30] have anything to do with the numbers that came out?

•TONY: Uh, yeah, they had like, uh everything.
Because, um, these [gears] are like, kind of like the measurement tools, sort of.
So wherever like they are, it's like the measurement, kind of.
[Dennis takes yellow dot stickers.]

•DC: Could any numbers come out?

•TONY: Yeah, it's possible.
Any numbers could come out.
[Dennis places a small dot on each gear wheel.]

•DC: If you came back and measured with this [gear set], like later, could you get different numbers?

•TONY: Well, it depends.
It's like, if somebody...
It's a matter of opinion.
Cause like if somebody, like changes it, yeah, it'll be different. Or like if somebody thinks it's one thing and then another person thinks it's another, that could change, like, to opinion.

•DC: Uh, huh. What do you think, Dennis?

•DENNIS: This [small gear] is 3 times...[unwinding divers]

•TONY: as fast

•DENNIS: ...as fast as the big one [gear wheel].
'Cause you can tell
•DC: |What, the small one? [gear wheel]

•DENNIS: The small one.
•TONY: |Yeah, this one. [pointing to small gear]

•DENNIS: 'Cause you can tell.
If you do it like this: [lines up both dots by turning gears]
Both [dots] are in the middle.
You go around once [turning gears until dot on small gear is back to where it began]
that's one time [pausing, then turning gears]
two times [pausing, then turning gears]
and then 3 times.
•TONY: |that's 3 times
[Dennis stops turning gears when both dots are lined up again.]

•DENNIS: So it's 3 times as fast.

•TONY: Yeah.

In between episodes 2 and 3, David asks to see the gears with dots on them be turned again. Tony shows him, counting the small wheel going around 3 times and saying that this proves that it's 3 times as fast. David then asks Tony and Dennis if they expect their data to be like the data James and Alex (2 other students in the class) had collected from the same gear set (A).

Episode 3 begins with David, Tony and Dennis looking at a table of numbers that another pair of boys (James and Alex) created FROM THE SAME GEAR SET (A). (See diagram 2). NOTE: James and Alex took their measurements by lowering the Right Diver (the faster one) instead of the Left Diver, by one unit at a time.

DIAGRAM 2

Left Diver	Right Diver
1/2	1
1	2
1.5	3
2.5	4
3	5
4	6
4.5	7
5	8
6	9
7	10

(3 minutes later) *Episode 3- Other Tables (4:46-6:40)*

•DC: Did you hear what he [Tony] said?

•DENNIS: No

•DC: He [Tony] said 3 times as fast. [pointing]
He [Tony] was talking about 6 and 9 [on James and Alex's table]

•TONY: Yeah. See right here? [pointing to 6,9].

•DC: Yeah, let's see.

•TONY: Yeah, because 6 and then 7, 8, 9.
That would be thr/.
So the right driver's faster.

•DENNIS: ( ) [talking with another student]

•DC: Is that 3 times as fast?

•TONY: Yeah. 6 and 9. 5 and 8. 4.5 and 7 [pointing to
numbers in the table].
The whole thing is thr/

The, always the right driver's 3 times as fast.

•DC: 3 times as fast.
What do you think, Dennis?

•TONY: Well, it has to...

•DC: Dennis?
Does this over here [pointing to 6,9] look like 3 times as fast to you? 6 and 9?

•DENNIS: Umm....No.

•DC: What's wrong?

•DENNIS: If it was 3 times as ...
What do you mean 3 times as fast? Where?

•TONY: Well, here. 'Cause see? [pointing to 6,9]
He's [left diver] at 6, he's [right diver] at 9.

•DENNIS: That's not 3 times as fast.

•TONY: It isn't? Well.

•DENNIS: It would/ That would be 18 if it was 3 times as fast.

•TONY: It wouldn't?/

•DENNIS: It's not even 2 times as fast.

•TONY: It isn't? Oh. Oops.

•DC: Well, he's [Tony] not so sure about that.

•DENNIS: If it was at, 2 times as fast, it would be 12.

•TONY: Oh yeah, OK, OK.
So it' would, like. 2 times.
Well, it's 2 times as fast I think.
Or less. No?
•DENNIS: |He's less than 2 and more than 1.
•TONY: |One times as fast.

•DENNIS: He's less than 2 and more than 1.

•TONY: So he's/

•DENNIS: [It's one and a half?
One and a half?
•TONY: [Yeah, it's one and a half.
•DENNIS: ['Cause half of 6 is 3.
And one,
•TONY: [and one.
•DENNIS: [And one times the 6 is 6.
•TONY: [So it would be...
•DENNIS: [And half of 6 is 3.
•TONY: [So it would have to be 9.
•DENNIS: [6+3 is 9.
So it's one and a half as.
•TONY: [Yeah, so I made a mistake.

•DC: One and a half.

•TONY: So I made a mistake.

•DC: You made a mistake?

•TONY: Yeah...

Episode 4 takes place 30 seconds later. Boys were searching for markers in that time period.

(30 seconds later) *Episode 4 - Gears and how many times as fast? (6:50-7:50)*

•DC: Can I ask you one more question?

•DENNIS: Yeah
•TONY: [Sure

•DC: Didn't you say before that these gears here, the little one, goes 3 times as fast as the large one?

•DENNIS: Yeah.

•TONY: Yeah, that's what we said.

•DC: Do you still believe that?

•TONY: Not entirely.

•DC: Why not?

•TONY: Well because we proved ourselves wr- wrong.
That, that was...

•DENNIS: It goes 3 inches faster.

•DC: 3 inches faster.
What about that story with the one and a half that you
were telling before.
Does that have anything to do with it?

•TONY: I don't think so.
•DENNIS: No, I don't think so...

•DC: Not one and a half times as fast.

•DENNIS: Yeah one and a half times as fast.
It is one and a half times as fast.

•TONY: Yeah

•DC: One and a half times as fast?
Or three, or 3 more?

•TONY: Well one and a half times fast but 3 inches faster.

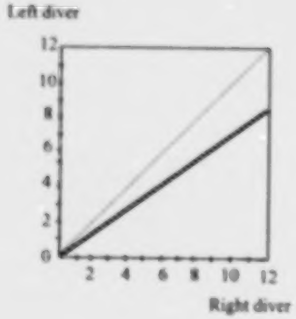
•DC: OK, great, thanks very much...

12/16/94-the next day

During class on the 16th, students were no longer working with gears. They'd begun working with colored, 1-inch square tiles in order to further investigate number patterns and graphs. The interaction in Episode 5 takes place with Tony and Dennis after class in an interview setting with Ricardo Nemirovsky and David Carraher (from behind the camera). Because they had not yet had a chance to create and explore graphs from the tables connected to the different gear sets, Ricardo and David took them after class to work on this.

The graph that they are working on had axes provided which were labeled, "Left Diver" and "Right Diver" and had a scale of 1-12 units of distance on each (see diagram 3). This is the first time the boys have seen a position vs. position graph. The graph they create shows the left diver's location relative to the right diver's location. The boys have been creating the graph shown on the tape by plotting the points from their table of numbers taken from Gear Set A (20:30). They are finishing the last point when we join them.

DIAGRAM 3



(the next day) *Episode 5- Graphing the divers (8:00-13:40)*

•DENNIS: They [the two lines] meet right here. [Dennis gestures a horizontal path at about 8 with his marker.]

•TONY: Right here. [Tony points to a spot at about 8,12 with his marker.]

•DENNIS: So it would be right there [pointing very close to Tony's spot with his marker].

•TONY: Yeah, like right here, OK. [drawing a point at 8 1/4, 12] Alright. Go. [Dennis draws a line connecting the previous graph line and the new point at 8 1/4, 12.]

•RN: So, looking at these lines can you see which one, which diver went faster or goes faster? [Dennis points to the 12 on the horizontal axis. Tony points to the same spot.]

•TONY: Um, yeah, because...

•RN: How do you know?

•TONY: Because according to this [points to vertical axis at 8], he's [Left Diver] only here [pointing to 8 1/4, 12] at the end, and he [Right Diver] is here [pointing to 8 1/4, 12].

•DENNIS: [pointing to 12 on the horizontal axis]

I have a clear reason.
Umm, because you can tell that this driver finished the race
before that driver. If they had tied/

•TONY: It would have been like,

•DENNIS: It would have been more [gestures with his hand in
the air near his face]

•TONY: It would have been, [pointing to point 12, 12 with one
hand and 12 on the horizontal axis with the other].
it would have been like up here
•DENNIS: [vertical [raising his hand above his head].

•TONY: Yeah, more vertical, if they had tied.
•DENNIS: [If they had tied,
[Dennis gestures a horizontal path from 12 on the vertical axis
to point 12,12.] it would have been right here [12, 12].
•TONY: [it would be more, more vertical.

•RN: What if you draw a line that shows/ [gestures a small
line]

•TONY: a tie?

•RN: a tie, right.

•TONY: OK.

•DENNIS: So [gesturing a horizontal path from 12 on the
vertical axis to point 12,12.]

•TONY: It would be umm...

•DENNIS: ...right here [draws point at 12,12].

•TONY: Yeah. And it'd be like,

•DENNIS: Can we use that yardstick?

•RN: Um hmm. [Dennis lines up the yardstick between 12, 12,
and 0, 0.]

•TONY: Let's use black [marker]. To show the difference
[between the 2 graph lines].
Got it? [the yardstick]

•DENNIS: Yep. [Tony draws the tie line.]

•TONY: That would have been about tied.

•DC: Where are they tied?

•TONY: On 12. [Both boys point to 12, 12.]

•DC: Have they tied anywhere else?

•TONY: Uhh, Yeah, they tied the whole way, I think. Yeah.

•RN: So being below this [tie] line [gesturing down and right]
what, what/

•TONY: Was like how the trip was.
This [pointing to 12,12] was like if it had been a tie.

•RN: And being like here [gesturing a line with a 60 degree
slope above the tie line], what does it mean?
•TONY: |Well, it's..
•RN: Which one is, goes faster if you would be, create a graph
that goes...

•TONY: Vertical?

•RN: More vertical, right [gesturing the 60 degree line].
•DENNIS: |And it goes...see these squares [blue lines on one
inch graph paper]?
It goes, point by point [pointing to blue corners intersecting along the tie line].

•TONY: By point by point.

•DENNIS: And since this [gesturing on their original line] is
uneven,
•RN: Um hm.
•DENNIS: it [the original line] goes between the things.
[corners of the squares of the graph paper they're drawing on]
It doesn't stop at a point.
•TONY: |yeah, but this one
But this one [pointing to 10, 10] goes like this.
This one just keeps going um like, like see? This.
Yeah, see like here? [pointing to 10, 10]
He's at 10 [vertical gesture] and um, he's at 10 here [horizontal
gesture], so that's a tie and then they keep doing that all the
way back [gesturing along the tie line towards 0]
and all the way up here [pointing to 12,12].

•RN: So suppose that there is another gear set that goes like
this [putting yardstick at a 60 degree slope].
Which diver would go faster?

•TONY: Umm, probably the left dr-

•DENNIS: The left diver.
Because at 12 it'd be...

•TONY: like uneven. [Dennis gestures a path vertically down
from the yardstick at about 7,12]
He would only be at um, ni- 7.
•DENNIS: |7. [Tony makes a dot at 7, 12]

•TONY: So, like similar to what we did.

•RN: And, say that/
•DENNIS: |No, no, not at 7.
[Dennis moves yardstick and gestures down vertically.]
'Cause/ Right there [pointing to 6 on horizontal axis].

•TONY: Oh then it'd be at like 6.
•DENNIS: |6

•TONY: Probably like similar to what we did too.
•DENNIS: |So, he'd [Left Diver] be
at twice as fast as this diver [Right Diver].

•TONY: Yeah, but according to our..
•DC: |Twice as fast?

•TONY: Yeah, twice as fast.
•DENNIS: |Yeah.

•DC: Why?

•DENNIS: Because he's [Right Diver] at 6,
and he's [Left Diver] already finished.
•TONY: |Yeah, so, yeah, he uh, he, uh he didn't.
He [Left Diver] finished and he's [Right Diver] only,
only halfway finished.

•RN: So, how would you describe the these 2 speeds [gesturing
to both axes] as the line goes like this? [moving yardstick
upward, one end fixed at 0,0]

What happens with the speed of the left diver and/?
•TONY: |The left driver keeps/
The left driver keeps getting, going faster and faster and faster until it's [yardstick/graph line] like straight [vertical], then he wins.

•RN: And as you go like that? [moving the yardstick to the tie line]

•TONY: The right driver keeps, well the left driver's losing speed.

•RN: Unh hunh. And then?

•TONY: Umm, then they're like about tie here and when
•DENNIS: |Oh, OK!
[Tony touches yardstick with both hands where yardstick stopped at tie line.]
TONY: it goes like that
[Ricardo moves yardstick below tie line, still fixed at 0,0]
The right driver's starting to win now.

•RN: Um hm.

•DC: What about the speed?
•DENNIS: |Oh!
•DC: Can I ask a question?
•TONY: |What?
•RN: |Well [gesturing to Dennis]

•DENNIS: [taking the yardstick] If you had made, um, close that off [placing yardstick at 12 vertically and gesturing a line], close this side off [placing yardstick at 12 horizontally and gesturing a line], you could tell if it was a tie

•TONY: Or not
•DENNIS: |if you went to the right point, you could, you could....

•RN: Do it, do it. [Ricardo places yardstick at 12 vertically.]

•TONY: OK. [Tony draws 1 vertical line at 12 and 1 horizontal line at 12, using the yardstick that Ricardo holds, starting from the axis at 12, meeting at 12,12]
There...And, it's like an arrow sh-, it's like an arrow showing/

•DENNIS: At the right point where 12 and 12 is, you could tell it was a tie

•TONY: |tie

•DENNIS: because it/ [Dennis gestures 2 new lines with his hand.]
•TONY: |Like, that's the way you could confirm it was a tie.

•RN: Um hm.

•DENNIS: And the more back [gesturing with 2 hands tilting above the tie line] it goes
•TONY: the, shows like the left
•DENNIS: |it shows the left driver's winning, and the more that way [gesturing with 2 hands tilting down] it goes

•TONY: the more right it goes

•DENNIS AND TONY: the more right it goes, it's like the right driver wins, and the more left it goes, it's the left driver wins.

•TONY: So it's just the opposite.

•RN: If/ Go ahead [to David].

•DC: Can you tell how fast either one goes from looking at this?

•TONY: Uh, yeah. It's like, you can tell/

•DC: Like, if you look at this tie place here [David points to the tie line]

•TONY: There? [pointing to tie line])

•DC: ...can you tell if that was a fast or a slow race?

•TONY: Uh, It was like the same speed race.
It was medium.
They were just like going at the same speed.
Cause I mean, it's not really...
You can't really tell
•DENNIS: |It was/
•RN: |Um hmm..
•TONY: |the speed.
Because it's not like if they're running or just walking...

PANEL PRESENTATION

Tony and Dennis

ANALYSIS OF CLASSROOM INTERACTION DISCOURSE FROM A VYGOTSKIAN PERSPECTIVE

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INTRODUCTION

The aim of this presentation is to analyse classroom interaction discourse in five episodes from a unit on *Investigation in Number, Data and Space. Patterns of Change: Tables and Graphs*. In this case, a complex interaction between visual tactile and verbal logical activity has been designed by adults. So the attention will not be limited to speech only. My analysis will be mainly focused on the *adult role* in the interaction. This choice is related to my research interest in studying the teacher role in classroom interaction and is consistent with Vygotsky's emphasis on the adult role in the zone of proximal development (ZPD). To analyse the adult role, I shall focus on the adult intentions, as they can be reconstructed from the available information. I distinguish between two different levels:

1) how adults set the stage, where interaction happens (macro-level);

2) how adults take actually part in the interaction (micro-level).

Two main issues are under scrutiny: looking for patterns (in the gear episodes) and interpreting position / position graphs. Before analysing actual interaction between boys and adults (micro-level), I shall discuss how adults are supposed to have designed activity at the macro-level.

MACRO-LEVEL ANALYSIS (HINTS)

LOOKING FOR PATTERNS (Episode 1 - 4)

Looking for patterns is the first important aim of the unit. It is founded on a pragmatic base: the two boys are requested to handle (according to some precise rules) gears and two-column tables. The *gears* are made with toy materials, but they allude to technological tools of everyday life. The presence of gears in technology is ancient: Aristotle himself is supposed to refer to toothed wheels in his book on *Mechanical Problems*, where he gives a qualitative description of a gear. Moreover gears were employed to build automatic machines and other mechanical devices. In the experimental display a vertical scale is added, where to measure the elongation of the string from which divers hang. A *two column table* completes the display. The complete display (together with the rules to handle it) is an example of *tool of semiotic mediation* (TSM Vygotsky), that inhibits the direct impulse to react (e.g. the production of only a qualitative description of gears). It introduces an auxiliary stimulus that facilitates the completion of the operation, up to the numerical modelization of proportionality. The adults are supposed to have set the stage in order to use the potentiality of the display as a TSM to control boys' behaviour from outside up to the statement of patterns.

The concept of pattern is a *scientific concept* (SC): in a Vygotskian perspective a SC is neither a natural development of everyday concept nor a matter of negotiation, but is acquired through instruction.

Data collection is the pragmatic base on which the whole activity is founded: it is done by means of eyes, speech (including numbers) and hands. According to Vygotsky, the unity of (visual) perception, speech and action generates, by internalisation, the internal visual field, where to produce mental experiments (e. g. guessing and testing a pattern).

As the adults probably knew in advance, Tony appears unable to perform the task of collecting data alone, because of his uncertain understanding of the task (he is always - 10 - wavering between left and right diver), of his incorrect execution (Dennis has to correct Tony several times - 25, 30, 64), of his lack of knowledge about numbers (Tony confounds whole numbers and decimal numbers - 17, 27) and their representations (Tony mistakes 8.4 for 8 and one fourth - 66), of his lack of acceptance of order convention (Tony reads 11,7 instead of 7,11). Moreover, Tony's attention is easily diverted by markers (70): he seems interested mainly in action (79). The joint activity of Dennis and Tony has probably be planned as an example of activity in the ZPD, where Dennis acts as a more capable peer.

INTERPRETING POSITION / POSITION GRAPHS (Episode 5)

Interpreting (position / position) graphs is the second important aim of the unit. On the one side, graphs represent point by point the number table (and the pattern, if any); on the other side they represents globally some property of the pattern (e.g. straight lines represent proportionality, the slope of a line represents a ratio, and so forth).

The boys have been accustomed to draw and read distance / time graphs, while considering trip problems. Both seem to master point by point representation while drawing the graph.

However distance / time graphs have a distinctive feature: they have globally a rich semantics, as they are a representation of speed (different slopes represent different speeds). On the contrary, position / position graphs modelize a comparison of the covered distances (i.e. of the speeds when the motion is uniform). Actually in the pragmatic base the motion is not uniform, as boys have to stop to write down numbers. Hence the semantics of such position / position graphs is not immediately related to the question: who is the winner. Left diver can win in a lot of different ways. What can be under boys control is: first, the distance by which he wins at the end, and, second, the distance in every point of the race. The natural interpretation is so based on a point-by-point reading, rather than on a global reading. The global reading of graphs with different slopes has to be forced in the interaction. We shall come to this point later.

The macro-level analysis that has been given above allows me to guess some of the adult intentions in designing this unit and the special kind of interaction between Dennis and Tony.

1) the intention to use the whole activity as a ZPD for Tony;

2) the intention to use the gear - table display as a TSM for stating patterns;

3) the intention to force the interpretation of the position / position graph.

MICRO-LEVEL ANALYSIS (HINTS)

I shall go quickly through the transcript (a detailed analysis is beyond the scope of this short presentation) in order to identify adult intentions, as they emerge from the on-the-spot interventions. I shall intentionally omit other types of analysis focused on either the personal senses that are built and eventually modified by the participants or the role of other semiotic tools (such as gestures) in the interaction. No analysis can cope with the process of interaction as a whole: even if every episode is a unique event, it can be read in many different ways, and every reading puts the same episode in a different perspective. Moreover this analysis is tentative: it does not take into account the adult introspection, that could have been considered if the adults had joined me in the analysis. However it will give elements to enlarge, reinforce or contrast the macro-level analysis.

In the following, for each phase, I shall interpret adult utterances by distinguishing:

Silence, when nothing is uttered;
Question, when a direct question is posed;
Call, when somebody is called to interact;
Mirror, when some utterance is repeated.
Rule, when a rule of interaction is stated.

For each of them I shall give my tentative interpretation for the adult intention on the base of the interaction.

EPISODE 1

Lines 1-79	Silence	Intention: *to avoid interference with boys' activity.*
Lines 81-84	Question	Intention: *to put a new problem: is there a pattern?*
Lines 85-89	Question	Intention: *to negotiate the meaning of pattern* ('what you see')
Lines 91-99		Tony offers his 'pattern' *ahead*
	Silence	Intention: *to avoid interference with boys' thinking.*
Lines 100	Silence	Intention: *to avoid evaluation.*

EPISODE 2

Lines 102-109	Question	Intention: *to negotiate the meaning of pattern* (through relationship between gears and number tables) Tony recalls the pragmatic base (measurement)
Lines 111-114	Question	Intention: *to negotiate the meaning of pattern* (pattern as function) Actually the function is bijective but, in spite of words, it is not what Tony means (see 120 below)
Lines 117-127	Question	Intention: *to know more about Tony's reasoning* or *to negotiate the meaning of pattern* (pattern as independent from the person) The 'pattern' *ahead* can actually be realised in many ways
	Silence	Intention: *to avoid evaluation..*
Lines 127	Call	Intention: *to try for sociocognitive conflict among peers*
Lines 129-154		Dennis shows his mental experiment and offers his 'pattern'

		3 times as fast
	Silence	Intention: *to avoid interference with boys' thinking*
Lines 155	Silence	Intention: *to avoid evaluation*
		EPISODE 3
Lines 157	Call	Intention: *to try for sociocognitive conflict among peers*
Lines 161-181	Mirror	Intention: *to make Tony express his reasoning*
Lines 182-186	Call	Intention: *to try for sociocognitive conflict among peers*
Li[illegible] 1[illegible]7-192	Questions	Intention: *to force the conflict by evaluation* (does it look like?...what's wrong?)
Lines 194-210	Silence	Intention: *to avoid interference with boys' thinking*
Lines 212	Call	Intention: *to encourage cooperation between boys* (Dennis has to convince Tony)
Lines 214-238	Silence	Intention: *to avoid interference with boys' thinking*
Lines 240	Mirror	Intention (?): *to accept the solution*
Lines 242-246	Question	Intention: *to know more about Tony's utterance*
		EPISODE 4
Lines 248-267	Questions	Intention: *to try for conflict between two solutions.*
Lines 269-289	Quest. / Mirr.	Intention: *to know more about boys' thinking.*
Lines 291	Rule	Intention: *to state an interaction rule:* (*great* refers to be willing to answer)
		EPISODE 5
Lines 294-305	Silence	The boys are drawing. Intention: *to avoid interference with boys' activity*
Lines 307-313	Questions	Intention: *to force global interpretation of that p/p graph*
Lines 315-338	Silence	Intention: *to avoid interference with boys' reasoning*
Lines 340-345	Questions	Intention: *to appropriate the tie problem posed by Dennis*
Lines 347-369	Silence	Intention: *to avoid interference with boys' activity*
Lines 371-377	Questions	Intention: *to force point-by-point interpretation of 'tie'*
Lines 379-393	Questions	Intention: *to force global interpret. of a whatever p/p graph* (without recalling the pragmatic base of gears)
Lines 394-413	Silence	Intention (?): *to avoid interference with boys' reasoning* the adult wishes a global interpretation while the boys insist on
a		point-by-point interpretation.
Lines 415-447	Question	Intention: *to force global interpret. of a whatever p/p graph* (the adult acts as if the relationship between gear and pattern has been explicitly stated)
Lines 449-455	Question	Intention: *to know more about boys' reasoning*
Lines 457-481	Questions	Intention: *to force global interpret. of a whatever p/p graph*
Lines 483-484	Question	Intention: *to put a new problem on speeds*
Lines 485-486	Rule	Intention: *to state an interaction rule.* (the adult asks the permission to ask a question)
Lines 489-510		The boys reinterpret the tie situation
	Silence	Intention: *to avoid interference with boys' activity*
Lines 512-525		The boys reinterpret the general situation
	Silence	Intention: *to avoid interference with boys' activity*
Lines 529-531	Questions	Intention: *to put again the general problem of speed*
Lines 533-548	Questions	Intention: *to relate the general problem to the tie situation*
Line 549	Silence	Intention: *to avoid evaluation.*

The micro-level analysis elicits some adult intentions:

α) the intention to state interaction rules that change the custom of a standard classroom;

β) the intention to put new problems or **to appropriate a problem that is posed by a boy;**

γ) the intention to negotiate the meaning of pattern;

δ) the intention to force point-by-point interpretation of p / p graphs;

ε) the intention to force global interpretation of p / p graphs;

ζ) the intention to foster peer interaction,

η) the intention to avoid interference, evaluation,

θ) the intention to know more about boys' reasoning.

COMPARISON BETWEEN MACRO-LEVEL AND MICRO-LEVEL ANALYSIS

1) The intention to use the whole activity as a ZPD for Tony.

This macro-level intention seems to be related to a micro-level intention:

ζ) the intention to foster peer interaction. As Dennis is a more capable peer he is expected to act as an expert guide in Tony's ZPD. Yet the distance between Dennis and Tony is too large in the episodes 1-4 to make cooperation effective. Moreover Dennis shows no intention to help Tony. At the beginning, he works alone while Tony is observing for most of the time. The effect is that the unity of (visual) perception, speech and action is realised for Dennis only. It is not a chance that a few lines later Dennis can produce the 'pattern' 3 times as fast by a mental experiment. Later in the Episode 3 Dennis is still working alone: he continues to speak to himself even when DC (212) invites him to convince Tony. At the end, in the Episode 4, the potential sociocognitive conflict is hidden by the fuzzy meaning of the term 'fast' (comparison of speeds or comparison of positions?).

2) The intention to use the gear - table display as a TSM for stating patterns;

This intention is not realised anywhere. When Tony misunderstands the meaning of pattern and when Dennis proposes the pattern *3 times as fast* in the Episode 2, no control on the device is done. When Tony proposes to read the pattern *3 times as fast* as the relationship between 5 and 8, 6 and 9, 4.5 and 7 (the whole parts) no control on the other pairs of the table is suggested (and not even on the meaning of considering only the whole part of numbers: what does it mean in the gear to cut a 'piece of elongation'?). Rather than using the device as a TSM, the adult prefers always to look for a conflict between Tony and Dennis. Yet the boys do not interact to each other effectively. Moreover, when Dennis reach a right solution by arguing (with an interesting sequence of elimination of conjectures) only on the number table, the solution is not related to action on gears. So we have two solutions with a different status: the solution *one and a half time as fast* is based on a single line of the number table; the solution *3 inches faster* is based on three (?) lines of the number table and on the pragmatic base of the experiment with gears, where the right diver is sometimes 3 inches beyond the left diver. The ambiguity is increased by the ambiguous everyday meaning of the word *fast*, to compare speeds or positions (as in *5 minutes fast*). The adult does not try to unravel the situation, by coming back to gears once more and by negotiating a meaning for *fast*. The effect is that Tony is lead to accept both solutions: *one and a half time as fast but 3 inches faster*, without seeing the conflict. I have not even information to claim that Dennis himself has solved the conflict between the two solutions (269-281).

3) The intention to interpret the position / position graph.

This macro-level intention is related to two micro-level intentions, that are present with different weight in the interaction.

δ) the intention to force point-by-point interpretation of p / p graphs. This intention is actually realised by the adult in only in one case (371-377) even if the point-by-point interpretation seems to be the true problem for the boys, as the semantics of the graph can be naturally reconstructed only by means of a point-by-point interpretation.

ε) the intention to force global interpretation of p / p graphs. This intention is clear in the whole Episode 5, also when the boys clearly show that they are going elsewhere (e.g. 394-413). However the boys do not relate the change of slope to a change of the gear set but to an increase of speed in the same race, that cannot actually be realised with gears. Tony is confused (461, 468, 479, 525) and seems to repeat only Dennis' utterances.

Some types of micro-level intentions have not yet been focused in the macro-level analysis:

α) The intention to state interaction rules that change the custom of a standard classroom;

This intention aims at avoiding the patterns of interaction that are well known in standard classroom, where only the teacher has the right to ask questions, even if s/he is supposed to know all answers and the pupils have the duty to answer.

β) The intention to put new problems or **to appropriate a problem that is posed by a boy;**

Posing new problems is a part of the teacher's task; in this case the adult is willing also to appropriate problems that can be posed by the boys too according to the non - standard rules of interaction.

θ) The intention to know more about boys' reasoning.

This is a widespread intention, when in non standard classrooms the adults try to make sense of pupils' utterances.

The complex of the three intentions that have been described above could be summed up as:

4) The intention to construct an educational setting where pupils can take the responsibility for their learning.

This intention is consistent with

η) The intention to avoid interference, evaluation and so forth.

As the η intention appears very pervasive, it may well represent the deepest beliefs of the adults who take part in the interaction. An implicit evidence is offered by

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γ) the intention to negotiate the meaning of pattern. We have two examples in the interaction between the adult and Tony: the Episode 1, when the term *pattern* is explained as *what do you see* and the Episode 2, when the question about a relationship between the gear and the number table is transformed in questions about number tables (111) and about the reproducibility of the experiment. In the first case, the recourse to an everyday metaphor has the effect of activating an everyday sense of pattern (*a description of the way in which elongations are arranged*) that is different from the scientific concept, in the second case, the limited (if any) pragmatic experience with the gear set does not allow Tony to relate gear and table to each other. But unfortunately the meaning of 'pattern' is still too fuzzy to be used.

TO START A DISCUSSION

The final table sums up the results of both macro-level and micro-level analyses of *adult intentions*.
Some comments must be done. My analysis was done in a Vygotskian perspective. But was actually the experiment designed and implemented in the same way? I believe that the design is consistent with a Vygotskian perspective, but I guess that the quality of interaction is not. For Vygotsky, the process of learning cannot be distinguished from the process of teaching to the extent that a single word (*obuchenie)* is used in Russian for both. Actually, in a Vygotskian perspective, the adults themselves would have taken part in the activity as expert guides in the ZPD for both boys (and not only as problem posers or observers); the adults would have used the potentiality of the devices as TSM; the adult would have not treated the concept of pattern (that seems to be a keyword of the whole unit) as an everyday concept.
It is possible to imagine different models for adult interventions in the same situation and to guess different developments of the process of teaching and learning.
The mental experiment is interesting and useful and is left to the reader.

ADULT INTENTIONS	
MACRO-LEVEL	**MICRO-LEVEL**
1) The intention to use the whole activity as a ZPD for Tony.	ζ) the intention to foster peer interaction;
2) The intention to use the gear - table display as a TSM for stating patterns.	?
3) The intention to interpret the position / position graph.	δ) the intention to force point-by-point interpretation of p / p graphs; ε) the intention to force global interpretation of p / p graphs;
4) The intention to construct an educational setting where pupils can take the responsibility for their learning.	α) the intention to state interaction rules that change the custom of a standard classroom; β) the intention to put new problems or to appropriate a problem that is posed by a boy; θ) the intention to know more about boys' reasoning; η) the intention to avoid interference, evaluation; γ) the intention to negotiate the meaning of pattern;

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Mediation by tools in the mathematics classroom

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Abstract

The concept of "tool mediation" is central to Vygotskian analyses of cognitive development (Wertsch, 1991). Individual growth and participation in sociocultural practices provide children with two general types of tools: signs (semiotic systems) and technical tools (instrumental artefacts). In school, students come across a wide variety of sign systems in mathematics (e.g., tabular and graphical representations of numerical patterns) as well as instrumental artefacts designed to engage the learner in practical actions with selected aspects of the physical world (e.g., devices with gears that move objects along a numbered track). This paper uses the notion of tool mediation to analyze the relationship between representations built by the students on paper and the use of physical devices in mathematics instruction. More specifically, I discuss the interactions between two fifth graders as they work on a physical system with gears of different sizes, and draw tables of values and graphs to represent and study numerical patterns.

INTRODUCTION[1]

Classroom use of physical devices (or more generally "concrete materials") is accepted by many mathematics teachers as good practice in mathematics instruction. The *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), for example, specifically recommends that: "To provide students with a lasting sense of number and number relationships, learning should be grounded in experience related to aspects of everyday life or to the *use of concrete materials designed to reflect underlying mathematical ideas*." (p. 87, emphasis added) The research on using such instruments in mathematics education, however, has been labeled by Thompson (1991) as "equivocal at best." (p. 1) Several studies have indeed yielded opposing outcomes regarding the role and effectiveness of concrete materials in mathematics instruction. While Fuson (1986), for instance, is very optimist about the positive effects of materials-based instruction, Resnick and Omanson (1987) are more speculative.

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[1] The Introduction and the section on Concluding Remarks of this paper draw on the work presented in Meira (in press).

In this paper, I discuss the use of "concrete materials" in mathematics education in the more general frame of tool mediation suggested by the Vygotskian approach. That is, rather than simply asking whether the use of concrete materials enhances individuals' cognitive efficiency and mathematical performance, I explore the question of how instructional artefacts and representational systems are actually used and transformed by students *in activity*. In order to discuss this question, I present below two contrasting views of the nature and function of mediational tools in mathematics learning: the epistemic fidelity view, and a cultural view of tool-use.

A cultural view of tool-use

The cultural view suggested here begins with the assumption that tools (such as physical devices and representational systems) are instruments of *access* to the knowledge, activities and practices of a community (Lave and Wenger, 1991).[2] The types of tools and forms of access existent within a practice are interrelated in intricate ways with the understandings that participants of the practice can construct. The question is then: How do people (e.g., students) interpret and make use of the tools (e.g., instrumental and representational) which are part of a cultural practice (e.g., the mathematics classroom)?

Wenger (1991) argues that, as the object of an a person's activity, an artefact or sign can support the construction of *fields of invisibility* — when users construct unproblematic interpretations of a tool and it is smoothly integrated into activities— and *fields of visibility* —when tools extend users' access to information and participation in a practice. An example from mathematics learning is given below that illustrates this dual nature (visibility / invisibility) of tools, and the dialectic relationship between tools and cultural practices.

In an investigation of children's discursive practices in several contexts, Walkerdine (1988) describes one study that focused on preschoolers' understanding of size relations and their use of terms such as *big*, *small*, *bigger*, *smaller*, *biggest* and *smallest* during a math lesson at school, a clinical interview, and at home. She observed a nursery teacher's use of *The Three Bears* story as a way to "contextualize" the study of relational terms. According to the analysis proposed above, the bear tale was to work as an instructional tool aimed at increasing children's access to mathematical information about size relations and participation in classroom discussions (i.e., creating a field of visibility), and at the same time be unobtrusive in the activity itself (i.e., creating a field of invisibility). Walkerdine was, however, struck by the way that the children reacted negatively to questions such as "*is daddy bear bigger than mummy bear?*,"[3] even though they could make correct size comparisons in many other tasks in the same lesson, during the clinical interviews, and at home. She argues that although in the school task the bear-family was intended to instantiate size differences only, the occurrence of relational terms at home was strongly associated with the parents' control of their children's behavior, in particular with the mothers' regulation of food consumption. The relational terms seemed to embody for the children unequivalent relations of power within their own families, and which were brought to bear during the classroom activity. Thus, by transforming a story about size relations (from an instructional perspective) into one about family relationships, the children developed a cultural interpretation of the bear-narrative. This cultural reading caused the balance visibility-invisibility in the classroom activity to emerge in unexpected ways (for the researcher, at least), for that which was made visible (i.e., family relationships) could not be accounted for solely in terms of the children's mathematical *knowledge* of size relations.

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[2] My use of the term activity relates to its meaning in Activity Theory (e.g., Leontiev, 1981): it refers to chains of actions (*practical and intellectual*) actually carried out by people, and which acquire meaning within specific sociocultural contexts. The term *practice* is used as defining of a social group that share activities, ways of doing and communicating things.

The epistemic fidelity view of tool-use

In contrast to the cultural view, traditional conceptions of tool-use are rather narrow in that they focus on *intrinsic qualities* of physical devices or representational systems, and on how those qualities might promote individual cognitive efficiency by enabling users to "see" underlying principles and relations through them. In this perspective, the fields of visibility created by activities with concrete materials are seen as pieces of unmediated information which are supposedly lying right underneath physical objects. Furthermore, tools are assumed to be more or less transparent depending on the quality of formal correspondences between tangible features of objects and a target knowledge domain as understood by experts. Resnick and Omanson's (1987) work on "mapping instruction," for example, complied with this view when it predicted that a step-by-step correspondence between subtraction with Dienes' blocks and written subtraction should allow students to understand procedures on written symbols in ways compatible with the meanings they constructed and understood in the blocks world. Roschelle (1990) points out that this view reduces the idea of tool mediation and tool-use to a measure of *epistemic fidelity*. The epistemic fidelity view focuses then on a empiricist interpretation of tools, seen as containers or conduits of meaning, without considering the cultural dimension of tool-use. This view has serious consequences for mathematics education practice and research, as it makes it difficult for one to appreciate the variety of meanings students negotiate and renegotiate during activity in a social and material setting.

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[3] In the story told the "daddy bear" is the biggest, the "mummy bear" is median sized, and the "baby bear is, of course, the smallest.

ANALYSIS

The analysis presented below is based on the videotaped interactions of two 5th graders as they work on/with a physical device with gears of different sizes (instantiating a 2:3 ratio), and build representations on paper as requested by two interviewers during a math class (see attached protocol). This section is divided in two parts. First, I will list and summarize the main episodes of the segment analyzed. Second, I will discuss these episodes in the light of the cultural view of tool-use presented above. The discussion will tap on the nature of the relationship between instrumental tools and sign systems, as well as on the mediating role of social interactions in the mathematics classroom.

The video segment can be divided in four episodes as follows:

Creating data (0:00-2:50, lines 1-80). We notice first that generating data from manipulation of physical devices depends on the material instruments available as well as on the language one chooses to register the data. The lesson's design required the students to read the position of three-dimensional objects (called divers) against a two-dimensional scale. As in the activity of professional scientists, this ability may depend on knowledge about what kind of data to expect.

Speaking mathematically (2:50-4:50, lines 81-155). The second episode is triggered by one of the interviewers, who asked the students questions about what they saw in a table of values they had just built. This episode is marked by distinct orientations in the students' strategies for building quantitative relationships. One of the students (Tony) draws on the table of values to construct an arithmetic relationship based on the growing differences between the divers' positions at the scale. The other student (Dennis) uses the device itself to mediate his talk about the relationship in terms of the relative "speed" of the gears.

Appropriating the work of others (5:00-8:00, lines 156-292). Dennis and Tony are then asked to analyze the data produced by another pair of students (James and Alex), who have been working on an equivalent set of gears. Based on James and Alex's table of values, Dennis and Tony revise their own work, producing what they understand to be a more adequate description of the relationship between the gears and the effects they have on the divers' positions at the scale.

Re-representing data (8:10-13:40, lines 293-550). Last, the students are requested to work on a graphical representation of the data produced in the first episode. At this stage, a change in the "tools of the trade" (from table to graphical representations) generates a renewed competence to talk about the physical setup.

Two related questions are discussed below on the basis of these episodes: (1) How do the students make their practical actions on the device mathematically accountable?; and (2) What is the relationship between the students' actions on/with the physical device and the actions that emerge from their work with tables and graphs?

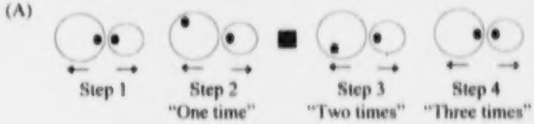
I start with the observation that the data produced by the students is not devoid of expectations such as: the data must somehow fit in a numerical pattern, or they will eventually be graphed. For instance, although the device was not designed for precision (e.g., the video shows many possible readings for most positions the divers are at on the scale) the data actually registered by the students reflect a relation where the position of both divers varies by constant rates (1 and 1.5 for the left and right divers, respectively). Of course, the students are not totally aware of the conceptual underpinnings of their data for the last pair in the table (8¼, 12) differ in significant ways from the others (perhaps because the scale ended at the 12th mark, as Tonny has noticed on line 98: "*when he's [left diver] at 8 and one fourth, he's [right diver] already finished.*") But it is clear that the students' activity at this stage cannot be reduced to merely reading off positions at the scale. In this sense, the data are not collected; they are rather created on the basis of the students' expectations about the behavior of a specific form of mathematical representation (tables of values). Tables are known to group numbers that reflect patterns that must somehow be made explicit in the classroom —so much so that Tony had a prompt answer to the interviewer's questions of "*is there a pattern here?*" (line 81), and "*what do you see?*" (line 85). As the product of the students' activity, the table develops a life of its own, in the sense that it is not a direct register of the behavior of the physical device.

Independently of the expectations the students might have held when creating the data, generating and communicating inferences on the basis of that data can be a difficult task. Indeed, the data as such may be "*a matter of opinion*" as Tony believed (line 122). That is, the quality of the data depends on factors such as the variable one takes to be independent, the exactness of the measures, one's expectations of what kind of data is possible, and the alike. Also, the activity takes place in a classroom where many such tables have been made publicly available through representations on paper. Perhaps for all or some of these reason, perhaps just because the device was there to be handled, the first attempt to relate the gears in terms of proportions (the task as given) was actually performed on/with the device.

In conducting an "experiment" with the device, Dennis marks facing dots on both gears and counts the turns of the 20 teeth gear until the dots face each other again (as in Figure 1A below). While the student concludes from this experiment that the 20 teeth gear is "*3 times as fast*" (line 152) as the 30 teeth gear, we see that his approach disregards the number of turns (2) completed by the bigger gear. During manipulation of the device, therefore, Dennis is unable to reflect upon a base unit to compare the gears, missing the important step represented in Figure 1B. It is also important to notice that the students seem to experience no conflict between the inference just mentioned, and the data as represented in their table of values. It is as if, at this point, the tools available (instrumental and symbolic) informed about two unrelated worlds.

After Dennis' suggestion of a 1:3 relationship between the gears ("*3 times as fast*"), the interviewer directs the students to inspect a table built by another pair of students on the basis of an equivalent gear set. Whereas Dennis and Tony's table did not adequately reflect the 2:3 ratio in any of the measures, James and Alex's table included three data pairs with the expected relationship (out of twelve measures taken). The interviewer begins this episode calling the students' attention to a "correct" pair previously mentioned by Tony (6, 9), and asking whether these numbers confirmed the suggested relationship of "3 times as fast" (lines 161-2).

While Tony seems to understand this relation as "3 units ahead" (lines 168-170), Dennis soon recognizes that the relationship should be "one and a half" (based on the observations that "one times the 6 is 6", "half of 6 is 3", and "6 plus 3 is 9") (lines 226-237).



(B) ■ Missing step: 1 time ⇔ 1½ times

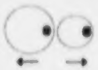


Figure 1: Dennis' experiment with the gears, and the "missing step".

We notice at this point that, while physical devices are generally used in instruction to provide a meaningful context for mathematics, it is the students' activity with mathematical representations that allowed them to understand the relationships embodied (by design) in the physical object. A major shift occurs in the students' activity. It is certain that actions on the objective (material) device contributed to initiate the whole thing, but it is the students' discursive activity based on mathematical representations that made them aware of the object itself. Furthermore, it is important to notice that this shift happened as the students, guided by an adult expert, appropriated the work of others into their own. That is, the mediation of tools during mathematical activity becomes itself mediated by social interactions. Finally, actions on the device were at first the source for creating the data, now the representation of the data is used to make visible the relationships intended by the lesson's designer.

Back to the site where the physical device stood, the students are led by the interviewer to think of the new relationship suggested on the basis of James and Alex's table ("*one and a half times as fast*"). Though Dennis and Tony perform no new test with the device, they seem to agree that both relations are valid if put in the right wording: "*3 inches faster*" (lines 269 and 289) and "*one and a half times as fast*" (lines 281-2 and 289).

The episode on graphing followed. There seems to be in this episode no attempt to integrate the information about the relationship of the gears (or of the number sequences) discussed in the previous episodes. The making of the graph allows instead a more general discussion about the divers' relative speed, based on the slope of the lines used to mark their positions. Here, the students' activity takes on a whole new set of ideas, influenced by expectations related to graphing. For instance, the reflection about slopes carried out does not require the exactness of the measures; the students build instead a holistic perspective of the divers' relative speeds in relation to a line representing a possible tie (slope equals 1). This activity represents, then, a considerable extension from the work carried out so far, in that the students begin to create possible worlds in discourse and through a new sign tool.

In sum, the following points stand out from the discussion above: (1) The analysis of students' activity with instrumental tools requires a parallel analysis of the sign tools that mediate their practical actions; (2) The representational and discursive practices that emerge in the students' activity with instrumental tools makes this activity accountable and meaningful for the students themselves;[4] and (3) Instrumental tools and sign tools may be said to constitute each other in activity, and the true dialectical relationship between them emerges in social interactions.

Finally, I suggest a shifting relationship in the balance between the visibility and invisibility of the instrumental tool available for Tony and Dennis during the episodes. The device was in one sense invisible for it functioned as a window through which the students could access mathematical forms of representing relationships. However, the device was sometimes visible and at the foreground of the students' activity, as in the episode where Dennis turned the gears with the dots until they faced each other for the second time (in a sense, reestablishing the initial, perceptually equilibrated state).

CONCLUDING REMARKS

One important theme of the analysis above was that the relationship between instrumental and semiotic tools is better not conceptualized as objective and determined *a priori*. The instructional quality of physical devices, for instance, relates to the very process of using them. That is, making sense of a physical device is a process that emerges anew in every specific context and is created in activity through specific forms of use.

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[4] An extensive analysis of the role of representational tools in mathematical activity is carried out in Meira (1995).

Moreover, instructional devices might themselves become the motive of much wondering and conversation in the context of which students engage in mathematical activity and argumentation. Walkerdine (1988), for example, showed that even when teachers set up lessons in terms of manipulation of physical objects, they continually use discursive strategies to reveal for the students the mathematical properties (e.g., of place-value) that the objects are supposed to supply. As in the segment analyzed above, it is the mathematical tools (e.g., tables of values) that are used to give the physical device particular meanings, not the other way around.

The analysis also contrasts the epistemic fidelity view with a cultural perspective. It indicates that designers of instruction and research may all too quickly assume the obviousness (or inadequacy) of instrumental tools as learning materials, while not paying enough attention to the contexts within which they are meant to function and the perspectives and histories of their users. Formal analyses of the inherent structure of physical devices can be partially enlightening to our understanding of artefacts-in-use. However, what the relation between people and instrumental artefacts *becomes* depends ultimately on their participation in specific practices, and on the practical and discursive activities in which artefacts are made to signify. For Lave and Wenger (1991), "the transparency of any technology always exists with respect to some purpose and is intricately tied to the cultural practice and social organization within which the technology is meant to function: it cannot be viewed as a feature of an artefact in itself but as something that is achieved through specific forms of participation, in which the technology fulfills a mediating function." (p. 102)

In contrast with the epistemic fidelity view, the perspective outlined here frames the problem of tool-use in terms of *mediated activity*. That is, artefacts become efficient, relevant and transparent through their use in specific activities, in the context of specific types of social interactions, and in relation to the transformations that they undergo in the hands of users (see also Latour, 1987). Furthermore, we should note that this sense-making process takes time and that even very familiar artefacts are not necessarily nor quickly well-integrated in the children's activities.

In whatever form, instrumental tools should have in the mathematics classroom an important role as *conversation pieces* (Roschelle, 1990), i.e., things which students can use to make mathematical ideas and arguments publicly available, and on the basis of which they can create their own material representations.

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PANEL "TONY and DENNIS"

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Episode 1

Tony and Dennis meet different kinds of problems:

- what is it to measure ? what is to be measured ?
- how does one express non-whole numbers ?
- what is the invariant relationship between the two sets of numbers ?

In this episode, Tony and Dennis are not tackling to measure speeds, but only lengths. One may even wonder whether they measure lengths, as the right diver was at ½ when the left diver was at 0. This is a problem all along the episode, which neither Tony nor Dennis are able to discover. One might even think that they express graduations and subgraduations rather then lengths. This is especially the case for Tony as he often gives a whole number instead of a decimal or a fraction in the first place (*6* instead of *6.5* on line 17, *9* instead of *9.5* on line 41, *12* instead of *12.5* on line 62). On line 66, he also says *8 point 4* instead of *8 and ¼* or *8 point 25*. For his defense, one must say that, according to the results I found 15 years ago, the division of the unit into 4 raises more difficulties than the division into 10. It is interesting also to notice that Dennis always expresses non whole numbers as fractions (*6 and a half* on line 19, *9 and a half* on line 44, *8 and one fourth* on line 68), Whereas Tony rather expresses them as decimals (*9 point 5* on line 46, *8 point 4* on line 66).

But the most striking moment is Tony's comment from line 91 to line 96. Tony tries to find an invariant additive relationship instead of an invariant ratio. As the differences are not the same when the divers go down, Tony can only discover that the difference increases.

The words used by David Carraher (*pattern, what do you see*) carry deliberately little information about the relationship to be found; they remain at the surface level.

Episodes 2, 3 and 4.

By pointing at the gear set A, and asking if it had anything to do with the numbers, David Carraher clearly performs a mediation act (lines 102 and 103). He changes the level of questioning. The problem is that we don't know if, beforehand, Tony has discovered something with the simpler gear sets (ratio 2 or 6).

Tony is obviously embarrassed. Therefore the next mediation act *"could any numbers come out?"* (line 111) is interpreted by Tony at the surface level of the sequence of ordered pairs. And Tony does not see any impossibility. He does not understand better than in episode 1, and considers that it is *"a matter of opinion"* (line 122).

Curiously, Dennis finds the ratio *"three times"* which Tony and Dennis interpret (correctly) as a ratio between two speeds (line 131). Dennis' explanation is not correct because he does not see that three turns of the small wheel correspond to two turns of the large one (lines 141 to 148). One can suppose he has done this kind of experiment with other sets of wheels.

The next table (page 5) is a perfect example that students have difficulties with fractions or decimals different from ½, 1½, 2½... All numbers in the left column are approximations by whole numbers or by whole numbers plus half.

The next mediation acts of David Carraher are important as they draw attention back to the ordered-pairs, more specifically to the case 6 and 9 (line 162). This is one of the two easy possible choices to find the correct ratio. It also conveys the wrong hypothesis *"three more"* . One more question by David is necessary for Dennis to realize that it cannot be *"three times as fast"* . He even finds arguments to convince Tony, and also finds that it is *"less than 2 and more than 1"* (line 220), and finally *"one and a half"*. Dennis explains correctly the reason Kath Hart had already found the comparative privilege of this value. Had the ratio been different, Dennis might have not found it.

By raising a contradiction with what had been said before (lines 253-254), David provokes something unexpected. Neither Dennis nor Tony start verifying anything with the set of wheels; it is Dennis that proposes an additive relationship *"three inches faster"* (line 269) and finally accepts the contradiction *"one and a half times fast, but three inches faster"*.

Episode 5

I find it quite remarkable that both Tony and Dennis, replying to Ricardo Nemirovsky's question, "*which diver went faster*?" point at the same spot 12 on the horizontal axis (lines 308 and 309). Yet, a short time later Tony is more confused, and points at (8¼, 12), to indicate the point where the left diver is "*at the end*"; also the right diver (lines 316 and 317). Dennis is less confused. Both Tony and Dennis agree that a tie would have been (12, 12) (lines 329 for Tony and 337 for Dennis). And Tony provides the comment "*it would be more vertical*". But it is Dennis that lines up the yardstick from (0, 0) to (12, 12).

David's question "where are they tied?" (line 371) entails a response which is too particular: (12, 12), but Tony adds rapidly "*they tied the whole way*". This again is quite remarkable, as 5th-graders do not usually understand graphs that well. The explanation of Tony (lines 406 to 413) is especially fine.

The answers concerning the other possible graphs (60 degrees) are also good. Dennis is even able to comment that 6 on the horizontal axis and 12 on the vertical axis is "*twice as fast*" (line 441). Tony is also able to interpret correctly the variation of the graph towards the vertical or towards the tie graph, or towards the horizontal. Both students are able to provide a very explicit interpretation "*the more right it goes, it's like the right diver wins, and the more left it goes, it's the left diver wins*" (lines 522 and 523). Some time later, Tony expresses the idea that all one knows about the "*tie graph*" is that "*they were just like going the same speed*" (line 542). As to the speed, "*you can't really tell*" (line 544).

Conclusion.

There are several conceptual problems involved in the sequence.

1) - invariance of the ratio between two distances, or two speeds;

2) - attribution of this invariance to the ratio between the number of teeth of the gear wheels, the circumference of the rods being the same;

3) - distinction between difference (additive) and ratio (multiplicative);

4) - approximation of fractions;

5) - control of the same departure points (0) for both divers;

6) - representation of the position of both divers by a graph;

7) - interpretation of the coordinates first, and later of the slopes of graphs, as indicating the faster diver;

8) - interpretation of the graph as a continuous invariant ratio between the speeds of the divers, and not as an increase or decrease of speed.

Some of these conceptual problems have not been touched at all: problems 2 and 5. Some problems seem to be difficult or at least delicate (for Tony more than for Dennis): problems 1 and 3 (and problem 4 most probably if one thinks of the table before line 156). The most striking positive fact is the very good approach, by both students, of problems 6, 7, and 8.

Also the careful mediation provided by David and Ricardo. Thank you.

RESEARCH FORA

The construction of algebraic knowledge: towards a socio-cultural theory and practice

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Abstract. *A theoretical model is sketched, which has been elaborated by the authors for analysing pupils' activities of production and manipulation of algebraic formulas. The model is based on the distinction between sense and denotation of an algebraic expression and on the notion of conceptual frame; these allow describing the way pupils attach a meaning to the algebraic formulas as well as their major misconceptions. Successively, the relationships between the model and the learning environments are examined and a detailed description is given of those features able to support school activities which produce a meaningful learning of algebra. In the end, a few examples are discussed.*

Introduction. Recent research in algebraic problem solving has identified as a key problem the relationship that students create between formulas and their meaning. The inadequacy of such a relationship often induces a void manipulation with symbols or an inefficient use of surrogates (see Sfard, 1991; Kieran, 1992 and 1994). There is general consensus that many students do not master the "sense" of those symbols which they have learned to handle formally. Sometimes they do not only ignore the meaning of formulas and concepts, but even arrive to invent meanings which surrogate the authentic ones. Other students, even if good "algebraic computers", do use algebra only as a computational machine and not as a tool apt to understand generalisations, to grasp structural connections and to argue in mathematics.

From a didactic point of view, it is very hard to overcome such misconceptions and difficulties, mainly because the invented meaning often has its own justification, frequently rooted in previously learned models. It may happen that the teacher and the student use the same words which correspond to very different meanings in their heads; a genuine comedy of errors is thus generated: the pupil and the teacher enter into a vicious circle, which is very difficult to break.

Existing literature has shown the possibility of taking instant pictures of students' difficulties but such a micro-analysis focuses on short-term phenomena and may be inadequate for studying long-term cognitive processes of pupils engaged in the learning of elementary algebra. In fact, the unbalance of time scales between pupils and teachers (or school) is fundamental for featuring the dialectic between learning and teaching the symbolic language of mathematics, especially the algebraic one: without such an approach, it is also difficult to elaborate suitable suggestions for teaching.

It is our concern to analyse algebraic thinking in the framework of a theoretical model we have elaborated on the ground of observed students' behaviours while solving problems. Starting from the consideration of students' difficulties, their typical behaviours and the evolution of their processes of thought, we outline a theoretical model apt to describe the main features of the cognitive dynamics occurring in solving algebraic problems. This model is intended to be the basis upon which an appropriate teaching-learning of algebra can be thought over.

First we approach the question of algebra as a language and a thinking tool and then we feature the environment where algebraic thinking finds its proper space.

1. An analysis of algebraic thinking

Processes of thought in algebra are recognised to be inseparable from its formalised language: algebra cannot be worked out without a written component. One must face the processes of teaching-learning of algebra considering both this written component with its syntactical rules and the processes of thought which make it possible (or not) for pupils to translate the oral strategies into written formulas. Generally, these are not a one to one translation of the sequence of mental operations into symbolic expressions (*stenographic function*) but express a reorganisation of the mental operations and of the oral strategies into the code of algebra (*ideographic function*, see Laborde, 1982).

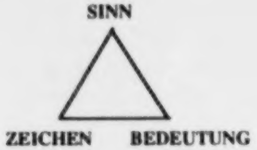
In accordance with Vygotsky, we consider algebraic thought and language as two intertwined and mutually dependent aspects of the same process. Analysing the word's central role, Vygotsky states that it would be incorrect considering thought and language as two independent activities, in spite of their separate origin. In particular, he stresses that the words' meaning is a linguistic and intellectual phenomenon which evolves in time. This statement is of special interest when applied to the meaning of algebraic expressions. In fact, it points out the *functional* aspects of that language, i.e. the language as a tool for *planning* and *anticipating*, because of one's needs and goals. Grasping the functionality of the sign-system of algebra (and more generally, of the symbolic language of mathematics) is one of the more delicate didactic problems of its teaching-learning.

As it has been pointed out by Land & Bishop (1966-69), see Mellin-Olsen (1987, p.53), two elements feature the way pupils appropriate of a sign-system functionality: the discovery of its own functionality (which depends on the familiarity and on the presence of an activity with it) and the decision by the individual to employ (or not to employ) the sign system (such a decision is usually social, that is made by the individual in relation to the others). Pupils will own the functionality of algebraic code insofar as they will be able to rule the relationships between that sign-system and its meaning; so a precise analysis of the nature of such relationships becomes crucial for understandings and planning didactic interventions in the classroom.

In this section we shall develop the following points: (1) a theoretical analysis of the meaning of symbolic expressions in algebra, with the aim of characterising the learning processes; (2) a description of the knowledge construction in algebra, considering also misconceptions and errors.

Let us start with a simple example: if you ask a pupil, who knows algebra, to split a number like 143, it is not usual that she/he uses the following strategy: $143 = (144-1) = (12^2 - 1^2) = (12+1)(12-1) = 13*11$. The expressions $/144-1/$, $/12^2 - 1^2/$ and $/13*11/$ all denote the same object (i.e. the number hundredfortythree) but show different ways through which such an object can be conceived, namely as a difference between two numbers, between two squares or as a product. The functionality of the algebraic code is concerned usually with the ruling of such productions (the above strategy in fact is fully algebraic, even if no letters appear).

A suitable analysis of the major ingredients of the algebraic code can be done using the ideas of Frege on semantics (see Frege, 1892a, 1892b, 1918). So we shall distinguish between Sinn (sense) and Bedeutung (reference, denotation, also meaning, but the English translations are ambiguous) of an expression (Zeichen): the Bedeutung of an expression is the object (Gegenstand) to which the expression refers (i.e. hundredfortythree in the above example), while the Sinn is the way which the object is given to the mind (i.e., as a difference of squares or as a product of numbers): see fig. 1.



Frege's semiotic triangle

fig.1

Mathematics and especially algebra is plenty of expressions whose senses are different but which have the same denotation. The most 'evident' sense of an algebraic expression represents concisely the very way by which the denoted object is obtained by means of the computational rules expressed in the formula itself; we call it the *algebraic sense*. For ex., the formula /n(n+1)/ in the universe of natural numbers expresses a computational rule, by which one gets the (denoted) set A = {0, 2, 6, ...}. But the same formula is able to incorporate additional senses, apart the algebraic one. In fact, it can be used in different knowledge domains, mathematical or not, each generating (at least) a new sense, depending on the nature of the domain. For ex., the expression /n(n+1)/ in elementary number theory has the sense of "product of two consecutive numbers", in elementary geometry it may stand for the area of a rectangle of (integer) sides n, (n+1). We call *contextualized sense of an expression* a sense which depends on the knowledge domain in which it lives (as such, it may be different from the algebraic sense): in fact the above formula expresses different thoughts, with respect to the different contexts where it is used (see Frege, 1918).

The power of algebra consists in the lots of senses which are incorporated by the same formula and/or which can be obtained by syntactic manipulations on it; whilst its didactic drama consists in the complete unbalance among senses, denotations and expressions, which makes the status of algebraic signs very obscure for students, because of their difficulties in grasping the functional aspect of the algebraic code.

To have a precise description of the dynamics of algebraic thinking, we still need an ingredient, that is the notion of *conceptual frame*. Let us make an example (see Arzarello et al., 1994b) .

> Consider the following problem (submitted to undergraduate students):
> "Prove that the number $(p-1)(q^2-1)/8$ is an even number, provided p and q are odd integers".
> We can see the following strategies in students:
> (1) Most students typically develop the formula as $(p-1)(q+1)(q-1)/8$ and argue considering the fact that factors are even, which is not enough for proving the claim.
> (2) Only a few, after some trials, change their mind, rename variables p and q and write down the following calculations: $(2h + 1 - 1)[(2k+1)^2 - 1]/8 = 2h \cdot 4k(k+1)/8 = hk(k+1)$.

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(3) It requires still some effort seeing that this is an even number, in the case k is odd, because in that case (k+1) is even.

In fact, all actions and decisions in the first strategy are ruled by students' knowledge concerning *even and odd numbers*: such a knowledge consists of an organised set of notions (i.e., mathematical objects, their properties, typical algorithms to use with them, usual arguing strategies in such a field of knowledge, etc.), which suggest them how to reason, manipulate formulas, anticipate results while coping with the problem, that is how to switch on and relate to each other the senses of formulas to be interpreted and to be manipulated in order to solve the problem. We call *conceptual frame* such an organised set of knowledge and possible behaviours. Conceptual frames are activated as virtual texts while interpreting a text, for example of a problem, according to context and circumstances; as such, they have socio-cultural and individual features.

We take the term *frame* from artificial intelligence studies (for ex., see Minsky (1975)); from this point of view, a frame is a structure of data that is able to get a stereotyped representation of a knowledge. However, our notion of frame is wider, insofar it entails also specific conceptual aspects of knowledge as an organised set of conceptual notions and operational skills related to some precise pieces of mathematics: so we call it a *conceptual frame*. As such, it is related also with the notion of *cadre* (setting), discussed in Douady (1986): the similarity with Douady's notion rests on the fact that a conceptual frame has also a strong mathematical dimension. The notion of conceptual frame is also close to *semantic fields*, introduced by Lins (1994), which in his view are a mode of producing meaning, a link between a statement-belief (a belief which is stated) and a justification for it.

Now we have all the ingredients necessary to define algebraic thought: in short, *making elementary algebra means playing a game of interpretations*: (i.e. activating different senses and/or producing different expressions in suitable conceptual frames) of a text in a semiotic system (for ex., a problem in ordinary language) into a text in another system (for ex., an equation), or from a text in a system (for ex., an algebraic expression) into a text in the same system (for ex., another algebraic expression). In fact, the interpretation is useful insofar as it makes possible to know something more about what is interpreted. The term "game" is in accordance with the way Kripke interprets Wittgenstein linguistic games (Kripke, 1982): there the paradox of the way a rule can be grasped is solved switching to the social dimension. This approach has important consequences also from a didactic point of view, insofar as it implies that a language to be taught, for ex. the algebraic one, should not be conceived as a pre-defined system of signs. In this case it would scarcely be interesting for teaching goals. On the contrary, it must be conceived as an activity with signs, which becomes a language through the linguistic game, hence acquiring a consensual and shared meaning.

When a student starts her/his interpretative activity, by various reasons she/he activates one or more frames, each connected with one Frege triangle at least. Which frame the student switches on, depends mainly on context, circumstances and connotations of the terms in the original text. *Once a frame is active, the student produces as a result of her/his interpretation a text, and the process of solution of the problem consists in successive transformations-interpretations of this text*, possibly in the production of completely new texts, according to the frames that are successively activated.

The game of interpretations can be actually made because of the two main functions of algebraic language, namely the *symbolic* and the *algorithmic* (in the terminology of Jakobson, 1956, pp.60-62); for more details and a less crude exposition see Cauty (1984) and Arzarello et al. (1994b). Using Frege triangle, such functions can be described easily. As a sample of the former, let us consider the processes of type (3) in our example: pupils can see that the formula k(k+1) denotes an even number also when k is odd; in this case the expression does not change, while its (contextualized) sense does (and possibly also its denotation). An example of the algorithmic function is given by the usual algebraic transformations, for example from k(k+1) to k^2+k: here the expression changes (and also its algebraic sense), but its denotation is invariant.

In short, the main point in our model is the game of interpretations, that is the dynamics among changing triangles; this process is called *semiosis* by some authors (see Duval, 1993): its main result is the production of *meaning*, namely the production of a *functional relationship among the senses and the denotations of algebraic expressions*.

The triangle of Frege, or its variants, and the relationships between the sense and the denotation of signs have been considered also by other authors working in mathematics education, with different meanings. For example, many people quote Ogden and Richards (1923): see Mellin-Olsen (1987) §2.1.3; the concept of *personal sense* or of *personal meaning* is a major concept in the Activity Theory: see Leont'ev (1994) and Schulz (1994). Without entering into a comparison, we stress that our triangles are dynamics, like in all the quoted examples. According to Frege the sense is not an idea or whatever in the mind of some people, how in some way all above authors seem to think. In fact processes of *semiosis* always have social features and their goal is to share with other people one's way of grasping an object. In other words, the subject makes it explicit so that it becomes part of the shared culture of a community. To quote Frege: "the sense does not constitute...something inseparable from the single individual, but can constitute the common ownership of many people. That things are so, is proved by the existence of thoughts which are common to the whole mankind and constitute an inheritance which is conveyed from generation to generation" (Frege, 1892b, §3).

As we shall point out later, for us the key point in the construction of algebraic knowledge is not only a question of conflict and of balancing in the subject's mind, but also a choral activity in which the participants (pupils and teacher) construct a socially shared algebraic language. This approach seems particularly suitable for breaking "vicious circle situations": in such cases it is the very interpretative activity of pupils in social interaction to enter the right tuning and to break the stall without any direct (and useless) intervention of the teacher. The main point is building up situations which stimulate the monitoring of the dynamics among expressions, senses and denotations (i.e. *semiosis* processes).

2. From the knowledge to its learning

The theoretical model described above is apt to interpret the nature of knowledge resulting from the use of algebraic language. However we have not yet concrete suggestions for designing suitable didactic situations, which allow pupils to build up a genuine algebraic knowledge, namely specific processes of *semiosis*, where the main functions of algebraic language can be ruled functionally by pupils. Hence we must analyse and describe how a specific piece of algebraic knowledge, which at the beginning of the teaching intervention is under the exclusive control of the teacher, has become under the control of students in the end.

To do that we shall introduce the following two main notions: (1) the *social space* of a subject; (2) the *didactic space-time of production and communication* (abbreviated with SP).

The former is taken from anthropology and has been elaborated by Fortes (see Cole, 1985, p. 153):

> "An individual's social space is a product of that segment of the social structure and that segment of the habitat with which he or she is in effective contact. To put it in another way, the social space is the society in its ecological setting seen from the individual's point of view. The individual creates his social space and is in turn formed by it. On the one hand, his range of experiences and behaviour are controlled by his social space, and on the other, everything he learns causes it to expand and become more differentiated. In the lifetime of the individual it changes *pari passu* with his psycho-physical and social development... In the evolution of an individual's social space we have a measure of his educational development." (Fortes, 1970, pp.27-28)

The importance of looking at the interplay between culture and cognitive development in cultural practices has grown up in the last decades after the pioneering papers of M.Cole and his colleagues at the Laboratory of Comparative Human Cognition (LCHC, 1978, 1979) and has already produced both theoretical frameworks suitable for analysing such relationships (for a survey, see Saxe, 1994) and concrete teaching projects for mathematics (Saxe, 1992). Such studies and projects focus on the relationships between individuals' goals in everyday practices and cognitive functions in their efforts to accomplish those goals. It is so possible to characterise the social space of a subject as the space where she/he realises a spontaneous learning through activities by which she/he can keep in touch with the meanings elaborated by the culture of the social groups with which she/he interacts.

Our goal now is to analyse the interplay between culture and cognitive development, in order to characterise processes of learning in one's social space with respect to mathematics; this interplay is delicate and interesting when approached for the specificity of algebra (as well as for many portions of mathematics, because of the pervasive role of its symbolic language) and will show the necessity of introducing the notion of didactic space-time of production and communication (SP).

The learning got in a social space is featured by the fact that the unbalance between the learner and the expert decreases and comes to an end during their interplay in a natural way. Namely, the times between learning and teaching are homogeneous: the social space of the subject enables both the learner and the teacher to share immediately the meaning of the system of signs used within an activity and concerning the taught knowledge.

To make us better understood, let us develop an analogy with the way children learn their own natural language. In case of speech, a natural feed-back is settled between the "teacher" and the learner and this guarantees them a real contact. The communication context, where their mutual relationships develop, allows them to share the same intentionality of thought and to experience the language's symbolic function of use. Things go differently with the learning of the written language; in this case, the child can only begin to get in touch with it, within her/his social space, for ex. by acknowledging the letters and some words, or by writing some simple words, etc.. However this is not enough to allow her/him a real learning of the written language. The aforementioned process of natural and spontaneous learning is not available any longer, without a systematic and well designed didactic intervention.

This is also the case of mathematics, particularly when its formal language is concerned: it is typical the case of written arithmetic and especially of algebra (which is not conceivable in an oral form). For ex., children can generally develop the ability of using a numeration system for counting sets of objects and also to solve simple problem situations. But, without a systematic and well designed intervention of the teacher, they cannot achieve the mastering of the positional system or of the symbolic function of arithmetic signs, as they must be used for solving problems.

There is an unbalance between the pre-scientific knowledge one can acquire spontaneously and the scientific knowledge that one must learn within explicit didactic interventions. Generally, such interventions are necessary when the unbalance between the expert and the novice in a certain knowledge domain cannot be resolved in a natural and spontaneous way. Of course, such an intervention must be built upon the dialectic relationships between the pre-scientific and the scientific forms of knowledge. The relevance of such a dialectic has been pointed out by Vygotsky, Bachelard, Brousseau and others, within different theories and with different consequences for the theories of learning. Also Bauersfeld points out a similar phenomenon when in the process of teaching he distinguishes the *matter meant*, the *matter taught*, the *matter learned*, that is the mathematical content which is planned to be taught, the content of the real teaching intervention and the cognitive structure of the learner after the process of teaching. It is our opinion that these three forms are strongly consonant and convergent in the processes of learning which develop spontaneously in the social space of the individual, but can diverge dramatically in those processes which need a systematic and well designed intervention, like the case of algebra.

We can now introduce the notion of SP (didactic space-time of production and communication). It will be used to picture the learning situations in algebra; in Chiappini and Bottino (1995a, 1995b) it has been used to analyse the learning situations in arithmetic. *It is a space (in the sense of the definition of Fortes quoted above) designed by the teacher in order to allow the learner to plan activities, because of a task.* The plan of the student has as its main goal the accomplishment of the task; the goal of the teacher is that the student gets in touch with the knowledge to be learnt, through the designed activity. In this frame, the learning of a knowledge can be developed only through an *activity*: the meaning attached to it depends always upon the functionality of the used semiotic system within the activity self. The word activity is here used according to the meaning that it has within Activity Theory and concerns the project developed by the learner according with the goals and the modalities she/he singles out of the given task within the SP. Hence the concrete process of learning consists in using the semiotic system's functionality within the SP, in order to incorporate the senses which are switched on and shared through the activity, achieving the meaning of the situation in the end. The word space is here used to mean the frame within which the subject's activity can develop as a *production-communication* activity. The former underlines the *intrapersonal* features, while the latter emphasises the *interpersonal* ones. An SP is defined by the possibilities of action that are available within it and by the specific features and modalities after which the subject's actions can be realised as production and communication activity. The SP supplies with the task, suggests the context and makes available the tools both for the action and for a socially shareable meaning of the action's product. By 'tools' we mean also the systems of semiotic representations, both as thinking and as communication tools. An SP is able to create a social space and is useful for the learning of the student if it can scaffold the activity of the pupil with respect to the knowledge to be learnt in a way that is accessible to the evoked social space. An SP allows to reproduce for didactic purposes (i.e. not in a spontaneous process) "the crucial match between a *support system* in the social environment and an *acquisition process* in the learner" (Bruner, 1985, p.28).

By *accessibility* we mean that the individual's social space must be able to control all the conditions which make it possible for the subject developing the designed actions within the SP. Interiorization of the activity made in the SP spreads out and differentiates further the social space of the subject and in the meanwhile causes exhaustion of its didactic function. Hence an SP is destined to come to an end: it goes out as a didactic support when the structure of the external activity has been interiorized, that is when the subject can reproduce such an activity at an internal level and is consequently able to incorporate the meaning which she/he has produced through the external activity in a suitable semiotic system of representation.

An SP is meaningful for the learning of knowledge, provided that the possibilities and the modalities of action, which it allows to the pupils, do realise the following goals: (1) motivating pupils for their planning of the activity; (2) supporting them in the choose of the specific goals and in the consequent processes of anticipation and planning; (3) emphasising the functionality of the system of signs into which the knowledge to be learnt must be incorporated, according to the design of the teacher.

The first two goals are a necessary, but not sufficient, condition for achieving the planned learning: they can be pursued by means of adjustment processes coached by the subject with respect to the possibilities and modalities of action within the SP. The third goal needs that pupils acknowledge such a functionality, and that they have a concrete accessibility to it within the SP: this means that pupils are proximal (in the sense of Vygotsky) to convert the represented knowledge from a known system of signs to another one and that the activity designed in the SP can support and help them in achieving such a goal.

Last but not least, the *time variable* is a parameter in a SP. Problems of time in the teaching of mathematics have been pointed out by Chevallard (1992) and by Brousseau (1991).

Time is crucial at least from two points of view. First, it is an *external variable*, within which the didactic space-time lives. It is important, because of the unbalance between the times of the teacher, and those of the learner, depending on her/his inner and social features. Second, time is an *inner variable*, inside the processes of pupils, for ex. inside their game of interpretation: related processes of internalization generally mark forms of contraction of the thought, whose most relevant aspect is the phenomenon that we call *condensation* (see Arzarello et al. (199?)): the word is taken from semiology (see Eco (1984), p. 157) and from Freud (1905); it has also some connection with the phenomena of *curtailing*, described in Krutetskii (1976), and with the property of *contracting*, typical of inner language (see Vygotsky (1934), chap. 7). Let us briefly sketch this last point. While making the game of interpretation, the stream of thought which sustains the computations and arguments of a student contracts its temporal, spatial and logical features and condenses them into an act of thought, which grasps the global situation as a whole. Such an inner process happens in a dialectic back and forth with the formula (such processes are clearly visible when pupils interact positively with spreadsheets, symbolic manipulators, etc.): there is a sort of converging process, from the formula to the subject and conversely towards the final solution, through possible different acts of thought and/or speech, which in the end culminates in a single act of thought in the pupil's head and in a general written formula (on the paper, on the PC screen, etc.).

3. Worked out examples

It is our claim that *effective learning situations in algebra can be developed only within a suitable SP*, so that the pupil can plan her/his activity developing gradually the meanings which are crucial for acknowledging the functionality of the system of signs of algebra. The last part of the paper is devoted to sketch a few examples which illustrate this point. The first three concern pre-algebra and elementary algebra, whilst the fourth (which is only sketched for space reasons) regards a more advanced topic of abstract algebra. They are taken from experiences made with students of different levels (from compulsory school to the University) in different places (Genova, Pavia, Torino). However, the subjects are not so relevant: our major point is illustrating the role of SP in describing processes of learning in algebra.

Example 1: the spreadsheet for the learning of early algebra.

We start with an example, that is the problem of naming a generic odd or even number (hence the frame is fixed). Such a problem is faced in grades 6-7 in Italy and we can introduce it within two possible SP.

SP1. Working with paper and pencil, the problem can be solved by the teacher introducing pupils to the expressions /2n+1/ or /2n/. Generally, the way by which such expressions incorporate the genuine sense of being odd or even is pursued by instantiating the variable /n/ with some numerical examples; this is done by the teacher at the blackboard and/or is given as an exercise to the pupils; with this aim also some numerical tables are built. In both cases the SP is designed to involve the pupil in a communication activity, which generally has all the features of a monologue, through which the teacher tries to convince the pupil about the power and capabilities of the system of signs that is introduced. According to teacher's purposes, the interpretation that the subject is supposed to give to the introduced signs should be based upon the meanings driven by the instantiation of the variable. However, the activity seems very poor and does not allow to many pupils grasping the functionality of use in the variable /n/; the interpretation of the expressions /2n+1/ and /2n/ given by the pupil can be done according with experiences and senses which are very different from those thought by the teacher. This can be the cause of a different interpretation with respect to the one socially shared. It is so produced a gap between the meaning conveyed by the teacher and that interpreted by the pupil. The "added value" of sense so got by pupils for the formula is generally very poor: average pupils who have done the above exercises show serious difficulties in producing-interpreting expressions where such a naming is required to solve concrete problems even at later levels of school (see examples in Arzarello et al.[94b]).

SP2. Now let us look at the above problem faced in a spread-sheet environment. In this case, pupils can interpret the algebraic expressions according with the aims of the teacher, albeit subjectively, provided they can experience their functionality of use within an activity that is apt to produce the right meaning for them. Pupils work with a sheet where in column A the sequence of natural numbers is generated by means of an iterative rule (fig.2). The column of values may be given by the teacher or worked out with pupils. After having named the numbers of column A by the letter N, pupils are requested to find a universal formula which generates odd numbers in column B using the numbers of column A as inputs. The pupils insert in the B2 cell the formula which solves the problem according to their conjectures; afterwards, looking at the column B of numbers so generated (which is produced automatically by the computer) they can check if the formula they have produced is adequate or not with respect to the posed problem .

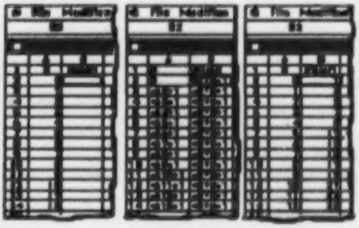

fig.2

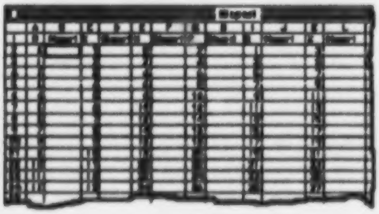

fig.3

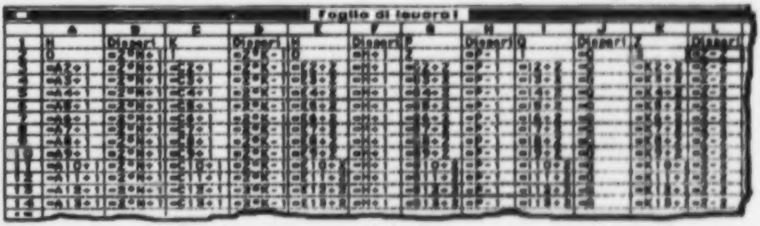

fig.4

In a further session, new sequences of numbers are given to the pupils and, in each case, they are requested to produce a universal formula which generates the sequence of odd numbers (fig. 3,4): for

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ex. /k-1/, /2k-1/,... represent an odd number if k is even, positive,.... Let us check which are the features of the SP designed by the teacher, that have made it possible for the pupils to plan a production activity, which has revealed meaningful for their learning. In SP2 formulas are constructed as a tool for transforming input data into results, according to the manipulative rules incorporated in the working logic of the software. In case of mistakes, pupils can try again, modifying the formula they have produced: a positive feed-back process is realised among the pupil, the computer and the formula self. Hence interactivity with software allows pupil to develop constructive processes which may be richer than those got working in an autonomous way in paper and pencil environments. In fact, *interaction* with computer in spreadsheet environments allows pupils producing and anticipating acts of thought, in order to grasp globally (some portions of) the formulas necessary for solving the problem. The numerical table got by means of the computer sustains and models anticipating processes of the subjects. They can so develop a common field of experiences, meanings, references, where they can construct suitable senses for formal expressions. The social construction of the meaning for formal expressions represents the most interesting aspect of experiences like that illustrated by SP2; emphasis is not on the computer as a performer but as a device which allows *interactivity*. In fact, such interactive aspectr may be relevant also in different environments, where computer is missing and the interaction is coached by the teacher using other constraints, for example organising properly internal and external group dynamics (see Cobb et al. [92] and our next examples), possibly forcing the use of suitable representations by pupils. Interactivity with the computer is a particular case of social interaction, where pupils negotiate the construction of meanings to attach to the products of their actions (for example formulas); computer is a cultural artefact (Saxe, 1992) that works as a *mediator* which constrains and sustains the negotiation self. When social interaction is missing (or less present) the pupils attach senses to symbols and formulas in a more private way, which may produce pitfalls and misconcepts. The result is that the subject is not able to use the sign system of algebra functionally with respect to the aims of the task. As an example, let us consider the following problem (Arzarello et al., 1994a):

> By means of a good choice of names to designate two subsequent odd numbers, show that their sum is a multiple of 4.

Experiments carried out with students attending Junior Secondary School (6-8 grades) and Undergraduate University Courses have revealed similar typologies of errors, namely:

- $x + y$
- $2h+1+2k+1$ or $2h+1+2k+1+2$ instead of $2h+1+2h+3$;
- moreover, a high percentage of students make only arithmetical checks.

The example shows that some students, who can express the relationships among the elements of the problem using the natural language or the arithmetic code, are unable to express them suitably by means of the algebraic code. More specifically, they are unable to use the algebraic code as a mediator between the identified goals of the problem and the qualitative and quantitative relationships among its elements.

Example 2: the learning of arithmetic with a pre-algebraic approach at the elementary school (gr. 4-5).

The example refers to a project for the teaching of mathematics in the elementary school; the project involve a few schools in Turin (about 150 pupils). Arithmetic is approached in a contextualized manner, for ex. simulating a market in the class, where pupils sell, buy, weight goods, use cash-registers, produce checks, etc. We have not the space for discussing all the activity and for giving a detailed description of its whole SP. We shall limit ourselves to illustrate an episode in a specific grade 4 class, which is useful for grasping how an SP can scaffold pupils' activities towards the construction of an algebraic formula.

Their previous knowledge concerning arithmetic and prealgebra is as follows: they are able to solve simple word problems where the four operations are involved (up to three, four operations in the same problem) and are acquainted in using letters for generalising problems (generally they use the metaphor of the machine:"how to give instruction to a machine if it must make the right calculations?" But none does attend any computer course). In the given situation, pupils are playing to be buyers and sellers in a market and are discussing the way retail prices for goods must charged, compared with wholesale prices. From their interviews with real sellers they know that taxes amount to 20% of final prices (in Italy when you buy a good, the seller tell you only the global price without distinguishing between VAT and the net price) and know how much further additional expenses affect the final price. In a discussion ruled by the teacher the full class has agreed about the facts that the final retail price must cover: wholesale price, taxes, additional expenses and the right gain for the seller. Now they produce collectively the following problem (the text is written at the blackboard by the teacher, according to the suggestions of the pupils):

> Which might be the price of pumpkin seeds, provided that the wholesale price is lire 8000 for a kilo, additional expenses can be evaluated in lire 1000 for each kilo and knowing that taxes are 20% of the total amount?

Pupils of the class (20 people) are divided into 5 small groups, each working autonomously and producing its proposal, which in the end is exposed to the the others and written at the blackboard.

1. Here are some example of the proposals of pupils:

Group of Valentino

8000 +
1000 +
20% =

9200
total amount

Group of Federica: **16000**

8000 + 1000 = 9000
16000 + 16000 = 32000
16000:10x2 = 3200
9000 + 3200 = 12200
16000 - 3200 = 2800

Group of Melania: **20000**

8000 + 1000 = 9000
20000:10x2 = 4000
9000 + 4000 = 13000
20000 - 13000 = 7000

2. Choral discussion

At this point the proposals are discussed by the whole class; social interaction among them clarify the mistakes of Valentino's group; but pupils are not satisfied by the solutions and try chorally other ones; after some attempts, they elaborate collectively a *method* which we call *by trials*, giving a list of examples as the following one:

"try with a global price of 18000 lire; the amount of taxes is 18000:10x2 = 3600 lire; that means 9000+3600 = 12600 lire as the global amount of expenses; hence the gain amounts to 18000 - 12600 = 5400 lire".

In the end they choose the example they like better among the different ones.

……………………………

3. Thinking again about it later on.

After two weeks the teacher recalls the problem to the class, distributes pupils a copy of the protocols they have produced in their discussion and invite them to discuss the protocols. They are again puzzled by the problem of taxes:

3.1. Fabio writes at the blackboard: "8000 + 1000 = 9000" and then : "20% of 9000 is 9000:10x2 = 1800" .
3.2. Teacher: Which will be the selling price?
3.3. Fabio writes: 9000 - 1800 = 7200.
3.4. Pamela: it is not right doing 9000 minus 1800; it gives too small a number.
3.5. Teacher: It seems to me that you are not sure that the selling price will be 7200 lire. Please make a check. Read again the text of the problem, remembering that now the selling price is 7200 lire and check if it fits with the other prices given in the text.
3.6. Federica: you must add 20% of the income.
3.7. Melania, Sonia and others: There is no income! It is missing.
3.8. Teacher: so?
3.9. Melania, Sonia: You must imagine it, think it before and then find the 20%.
3.10. Melania: and then you must add it to the other expenses.
3.11. Fabio: How can we add it? Excuse me...generally you take off! It is a number which becomes smaller, making this 20%.
3.12. Federica: It is true that it is a smaller number, but you must not take it off from expenses! You must add it to get the total expense.

4 . Pupils agree that the approach of Melania, Sonia, Federica is right, write down the procedure and with the support of the teacher produce formulas for solving the problem in a general way, namely:
4.1. PRICE:10x2 + (8000+1000) + GAIN = PRICE.

They also generalise the formula for different wholesale prices, for example 5000 lire (they explicitly say that additional expenses do not change):
4.2. PRICE:10x2 + (5000+1000) + GAIN = PRICE.

Let us discuss sketchily the functionality of signs within the given situation: there is an evident evolution from the productions of episodes 1 to the choral elaboration 2, up to the discussion in episodes 3.1-3.12. In the first episodes the productions are completely inside the arithmetic register; but the signs of arithmetic become more and more functional with respect to the generality (and difficulty) of the problem: arithmetic changes its status and its formulas are produced more and more according with an algebraic code. Pupils acknowledge this during the choral discussion in 2: the method by trial is in fact a set of arithmetic formulas whose sense is different from that of the formulas produced in episodes of 1 (even if they are the same arithmetic expressions). Pupils use the symbolic function of algebraic code to elaborate this new sense attached to the old expressions. Of course, in this process, functionality is accessible to pupils because of the meaningful situation where the problem is contextualised. It supports their activity, facilitating their anticipation of the very algebraic problem (note that the pupils themselves read the problem in an increasing algebraic manner).

As to the variable time, there are two important observations. The first is about the outer time and the relationship between the time of the teacher and that of the pupils: it is interesting observing that the jumps of pupils towards generalisation happen in episodes 2 and 3.6. Both are marked by the possibility for pupils to come back to their written productions (the first time immediately after the production, the second time after a fortnight). This fact is typical: we conjecture that the possibility of seeing "what one has thought" a second time in a written form allows people to break the constraints of sequential acts of thought in the time and makes accessible at the same moment things which were not so at the beginning.

The second observation concerns inner time: episode 3.9 is crucial for inventing the price as an unknown variable. Melania and Sonia widen their time horizon, get detached from previous productions and elaborate a new sense for the price and for the formulas that are under their eyes, namely the imagined price; then they can imagine to develop their calculations in the time. They make accessible their act of thought to other people by acts of speech, which tune their production with other pupils' thought (3.9: "you must imagine it"; 3.10:"and then you must add it to the other expenses"). At this point, with the support of the teacher, their acts can be condensed in a formula (episodes 4.1, 4.2).

Example 3: exploiting the isometry groups of regular polyedra (sketch).

It is an activity that I do with my students in a course for future teachers of mathematics. They learn concretely the art of paper folding (Japanese origami) to construct geometrical shapes, for ex. the five platonic solids. Then we explore together the main features of the groups of isometries for such figures; particularly they face the problem of representing such groups as (sub)groups of suitable permutation groups. The SP is given here by the relationships between students' activities with such artefacts and their knowledge about groups (which are very formal and poor: they do not own the concept of group as a tool for solving problems). Like in the case discussed in ex. 2, they first work in small groups of three-four, then each group presents its own solution and the choral discussion starts; after some weeks we come back to their productions and there is a second discussion. It is interesting to observe the slow evolution of their way of interacting among them and with the taught matter within the SP, while the didactic time goes on according with their own times. The major point is that the SP scaffolds their activities so that from the informal discussions about the objects they go over to a more formal level and use the formalism of groups as a tool to produce general proofs. The SP incorporates inside the germs for its own end as a didactic situation; this corresponds to the passing of students from interactions with the objects to interactions with a general meaning they have been able to attach to the SP activities.

Some final remarks

Let us discuss more specifically the nature of the support that an SP can give to the developing of pupil's knowledge in the case of the algebraic domain. Main algebraic activities concern producing-transforming-interpreting algebraic expressions and among the major processes which are essential for coaching such activities we find *anticipation* and *planning*, that is processes which are not always ruled in an autonomous way by pupils. The former concerns the activation of suitable senses for the algebraic symbols and expressions which concern the problem to solve. The latter may have two different aspects: first, in the case of the *construction* of a formula, it concerns the setting of the resolution process after a hierarchy of aims so that one can achieve the goal of the problem; second, in the case of the *interpretation* of a formula, it concerns the rebuilding of the hierarchy which was in the mind of the people who produced the formula self. In the construction of a formula, anticipation shows a back-and-forth stream of thought from inner processes to external representations, which goes towards a goal by means of a suitable planning of the resolution process. In the case of transformation of formulas, anticipation and planning concern the gradual construction of the shape for a possible final state of the expression, again in a feed-back process between thought and representations. In both cases the goal is a powerful stirrer of anticipation for the senses that the expressions must incorporate in order to fulfil it. A major didactic problem is to find suitable mediations for supporting anticipation and planning. Our claim is that such processes can be developed only within a suitable SP, where students can find both social interaction, that is interpersonal exchanges between the pupil and the environment (teacher, mates,...) and suitable mediators (namely cultural artefacts, like books, computer,...) which are aimed at producing meanings. To be useful for learning, activities in SP must allow pupils validating and justifying the sense of the expressions within her/his social space, that is within the rules and the culture that the subject effectively owns. All this constitutes the socially shared, even if not explicit, background which belongs to the social space of the pupils and constitutes the necessary basis for what pupils do and for their interpretations of what they have done. SP must be able to scaffold and support the student's activity; the learning process can be considered as an imitation, on part of the novice, of the expert's performances: from her/his side, expert can support the novice explicating her/his own strategies, streams of thought, skipped difficulties, while solving a problem. The social aspects of interactions are essential to scaffold pupils' activity as a *cognitive apprenticeship* and not only as a practical one. The novice can so learn algebra while working in a cultural and social environment which resembles the *Italian bottega d'arte* in Renaissance, or the *Samba School in Rio de Janeiro* (see the description given by S.Papert, 1980): in such *teaching-learning environments* the novice social space grows up also because she/he learns by doing, by seeing it done and by discussing systematically what she/he is doing both with experts and with other novices. Our present research is aimed at studying long-term processes, which tipically feature the evolution of pupils in the SP's, exploiting the analogy with the above natural models of teaching-learning environments.

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Students' Knowledge of Algebra

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A world-wide reform of the algebra curriculum is going on nowadays. Within circles of researchers, like at PME-conferences, an increase in profundity of the reflection on the subject-matter can be observed. However, much of the research on learning and teaching of algebra takes the traditional curriculum as a given. The research reported on in this paper does not take the existing algebra curriculum as a starting point , and as a consequence, the nature of algebra as we will discuss deviates from such research outcomes (see also Kieran, 1994). In this paper we try to move beyond the rhetoric on algebra reform, and we will describe some alternatives for the traditional approach toward learning algebra. We attempt to apply the key-ideas of our theory of mathematics instruction -- Realistic Mathematics Education -- to the research methodology we use -- Developmental Research. The theory and the research methodology are illustrated with examples from a developmental research study on "solving systems of equations".

1 INTRODUCTION

The place and role of algebra in school mathematics are under intense review on many fronts. The reasons for reform have been many and varied, and they are true for many of the traditional mathematics courses: The society has changed drastically (people need to be well educated to be able to deal with increasingly complex situations); Technology has taken over many of the cumbersome computational activities (and it has had its influence on the content of mathematics); New theories on learning and teaching (influenced by the constructivist approach) are replacing the old ones (the behaviorist approach); And, consequently, the goals of mathematics education are changing (there is more emphasis on higher level skills like problem solving, reasoning, communication, critical attitude, flexibility, making connections).

In the traditional algebra curriculum in the United States, algebra is presented as a language and a fixed structure. Students learn to copy the rules and tricks of algebra without a real understanding of the matter. The pedagogy is top-down, and too little attention is paid to the generalizing aspect of algebra, and to the dynamic aspects of variables -- that is algebraic reasoning. The jump to the formal level is made too quickly, and there is no time for students to develop their own schemes. The traditional algebra course is seen as sterile, disconnected from other mathematics and from the "real world" (Romberg & Spence 1993).

A large and growing body of research began to elaborate the cognitive underpinnings of algebraic understanding (see e.g. Booth 1984, 1988, Kieran 1992, Schoenfeld 1987, Wagner & Kieran 1989). Algebraic reasoning in its many forms, and the use of algebraic representations -- including graphs, tables, and formulas -- are very powerful intellectual tools that should be made available to all students (see e.g. Janvier 1978). In a recent curriculum innovation project in The Netherlands, this vision on algebra was used to develop teaching and learning materials for the age group of 12 to 16 years (see Team W12-16, 1992). Unlike many of the traditional research projects, this project was an example of integrated research and development. The project integrated all issues related to curriculum reform: developing curricular materials, teacher support and education, assessment, and implementation. The research methodology of projects like the 'W12-16 project' is very much related to the philosophy on mathematics education as developed at the Freudenthal Institute over the past 25 years. Before we describe in more detail some ideas and results of recent research on early algebra, we first explain the theory, or philosophy, underlying the work of the Freudenthal Institute.

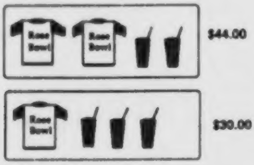
Realistic Mathematics Education

In Realistic Mathematics Education (RME), mathematics is considered as a human activity (Freudenthal, 1991). The main idea is to develop sense making ways for leaning and teaching sense making mathematics. The approach of RME was propagated and elaborated by Freudenthal and staff of the Freudenthal Institute (Treffers 1987, De Lange 1987, Streefland 1993, Gravemeijer 1994). We will describe RME using the following characteristic features.

'real' world

In the teaching and learning of mathematics, realistic contexts (or situations) play a crucial part: The for the students experientially real world is used as both a meaningful base to start the development of mathematical concepts and skills, and as a field to apply these mathematical concepts and skills. The contexts -- or problem situations -- need to be authentic and realistic to the students. This doesn't mean they need to stem directly from everyday live. The contexts should make sense to the students, and they must support the students in solving the problems and developing (or reinventing) mathematics. The context can also be mathematics itself, mathematics can be realistic as well.

Spending Money at the Game



How much does a T-shirt cost?
And how much is a soda? Explain how you got your answers.

A Birthday Party

Mom, Dad, and Rachel are celebrating a birthday at the dinner table.
Rachel says: "Hey Mom and Dad, your ages total 100 years now!"
Dad is surprised. "Yes you are right. And your age and mine total 64 years."
Rachel replies, "And my age and Mom's total 58."

1. How old are Mom, Dad, and Rachel?
Explain How you got your answer.

Figure 1. Some examples of 'realistic' problems, related to 'solving systems of equations'.

Figure 1 shows two examples of problems to be considered 'realistic'. Both problems involve the same mathematics, but use a different representation. The problems are 'real' in the sense that they make sense to the students; Students are able to realize what the situation is about.

We will come back to these problems to show how the context supports students' reasoning in solving the problems.

own productions and constructions

In RME we often start with a problem situation that students are familiar with. The existing knowledge of the students that they bring in, is used as a starting point. We build on this existing frame of knowledge and reference. It is important to identify if all students are at the same thinking level, and if they have the prerequisite knowledge before new mathematics is developed. Students' own productions and constructions are good means to find out and to establish a common starting point.

Moreover, students' own productions and constructions are also used to reflect on the teaching-learning process, in order to help students to progress to a higher cognitive level of doing mathematics.

Streefland (1995) carried out a small experiment to investigate the possibilities of a new approach toward teaching and learning 'solving systems of equations'. Students were asked to compose little bags of sweets and determining the prices of these bags. At first, the prices of the sweets were given, and the students were asked to develop methods to compare the content and the prices of the different bags. Next students imagined to be the shopkeeper, and they were asked to make combinations of sweets like 'two chocolate and 5 liquorice lace cost 2.05'. Students came up with a rich variety of notations and methods to describe the bags of sweets. These own productions and constructions were then used to discuss what notation systems were efficient, and under what conditions it is possible to find the prices of the sweets.

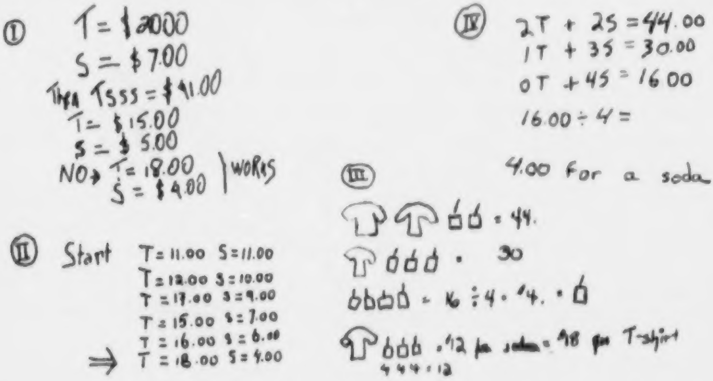
mathematization

A first step of problem solving is interpreting the problem situation and to translate it in some kind of mathematics. This process is called horizontal mathematization. By allowing students to use a variety of informal and pre-formal methods at different levels of abstraction and formalization, all students can solve problems using mathematics that makes sense to them, and that they feel comfortable with. Through reflection using own productions and constructions, students are encouraged to make the step from a lower cognitive concrete level to a higher level of abstraction. This formalizing aspect of RME is labeled as vertical mathematization (Treffers, 1987). The challenge for the researcher/developer is to find problem situations that are paradigmatic for the intended learning process. Students can develop a (thinking) model *of* the problem situation, and use this model *for* solving mathematically similar or related problems.

The teacher and the instructional materials play an important role in mathematizing. They guide the students in reinventing mathematics. The pedagogy is that of starting with an open learning environment from which the teacher and the curricular materials facilitate and guide -- through reflection and discussion -- the learning process of the students.

In algebra instruction, mathematization means that students learn to algebraize, that is 'to make algebra of the problem situation'. Generalizing the individual strategies of the students helps them move from the informal level to the formal level of doing algebra. This process can be characterized as progressive formalization.

For example, the methods and strategies that the students came up with in the experiment of Streefland were discussed and reflected upon in class. The goal of the reflection was to find ways to generalize and formalize the methods of the students.



I
T = $20.00
S = $7.00
Then TSSS = $41.00
T = $15.00
S = $5.00
NO
T = 18.00
S = $4.00
WORKS
II
Start
T = 11.00 S = 11.00
T = 12.00 S = 10.00
T = 13.00 S = 9.00
T = 15.00 S = 7.00
T = 16.00 S = 6.00
T = 18.00 S = 4.00
III
= 44.
= 30
= 16 ÷ 4 = 4.
= 12 for soda = 48 per T-shirt
4 4 4 = 12
IV
2T + 2S = 44.00
1T + 3S = 30.00
0T + 4S = 16.00
16.00 ÷ 4 =
4.00 for a soda

Figure 2. 'Guess & check' strategies (I and II), and 'reasoning' strategies (III and IV).

The two problems presented in figure 1 can also be solved on different levels of mathematical sophistication. One can use a random guess and check strategy, a reasoning strategy using the idea of exchange, or algebraic equations. Figure 2 shows some examples of 'guess & check' and 'reasoning' strategies for the T-shirt problem. The task of the researcher/developer is to design a natural environment in which the students will develop in a sense making way the more formal concepts related to 'solving systems of equations'. We will come back to the use of different strategies later in this paper.

interaction

As indicated in the former paragraph, reflection and discussion are critical factors of the teaching-learning process. The own productions and constructions are subject for discussion, organization, and reflection. In RME, interaction between students-students, teachers-students, and teachers-teachers are all important ingredients. This is paying much lip-service to the latest fashion in (mathematics) education. However, this does not mean that group work is the best and only form of instruction. Working in cooperative groups can contribute to learning, but it is up to the teacher to decide what the most appropriate form of instruction is for her students and for the situation in her class. Interaction is important because it leads to crucial activities of problem solving: cooperation, discussion, sharing and reflection.

integrated learning strands

In the philosophy of RME, school mathematics is a subject in which different mathematical topics are integrated; There are no separate courses algebra, geometry, and statistics, calculus and so on. Because instruction is not hierarchically organized, all students get the opportunity to learn about all topics of mathematics. Traditional instruction -- especially in the middle grades -- often takes a linear approach, and it beliefs in repetition: A topic is introduced, taught, and practiced, all in a short period of time in one unit of instruction; Students are expected to 'master' this topic by means of reproduction, and the topic is not revisited until probably the next year in the same chapter of the book. We know from experience and from research that this approach does not contribute very much to learning. In RME, mathematics is seen as an integrated subject, that is developed as a whole, and the connections between the different sub-domains are constantly made explicit because these connections are implicitly present in the materials.

some remarks on RME

One of the causes for malfunctioning instruction is the discrepancy between the actor's point of view and the observer's point of view. The actor -- or student -- has not constructed and developed the knowledge of mathematics as the observer -- the teacher -- has. The teacher, or expert, uses a different frame of reference than the student, and that causes problems in communication and mutual understanding. This discrepancy is the result of a top-down approach in instruction in which mathematics is presented as a fixed body of knowledge that needs to be mastered and that can only be imitated by reproduction. Realistic Mathematics Education is a bottom-up approach in which students' own productions and constructions in their own language and notation systems are used to develop the mathematics.

RME is more than a view on mathematics education, it also includes a vision on research and development, and it also reflects the attitude toward reform of mathematics education. In the philosophy of RME, learning, teaching, assessment, curriculum development, and other aspects of instruction are all related. They mutually influence each other, and therefore, all these issues should be addressed at the same time in a mathematics education reform project. Thus RME involves educational development at all existing levels.

2 DEVELOPMENTAL RESEARCH

In the former paragraph we described RME using five characteristic features. This paragraph is an attempt to describe the research methodology using the same five characteristics. The work at the Freudenthal Institute is mostly a combination of research and development, actually research and development are integrated. Characteristic for 'developmental research' are iterative research and development cycles, with development efforts supported by the research, and with renewed research efforts triggered by issues that arise in field experiments. Both the instructional materials and the underlying theory constantly evolve during this cyclic process.

real world

The school, class, students and teachers are the reality from which the ideas about mathematics education originate. The researcher/developer has strong connections with the daily practice of teaching and learning. By teaching, observing, and visiting classrooms on a regular base, we keep a feeling of what really happens in the classroom.

After an idea -- or theory -- on new ways of teaching and learning mathematics is 'born', it is materialized in instructional materials that are tested in the classroom. So, the classroom plays the role of a meaningful source for new ideas, and as a place to find out if the new ideas are feasible, and if they work.

own productions and constructions

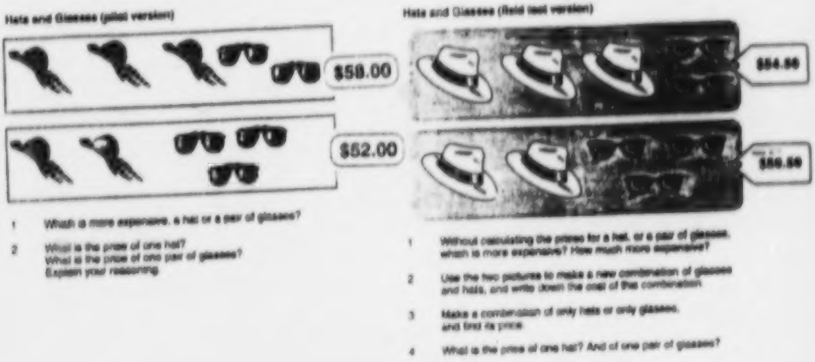
As students' own productions and constructions are important for the teaching and learning of mathematics, so are the researcher/developer's first materializations of ideas embedded in experience, common sense, and existing research. As described earlier, the task for the designer/researcher is to find paradigmatic problem situations for the development of the mathematical concepts. In order to realize an alternative approach for learning some particular mathematical topic, it is essential to carry out a thorough analysis of the subject area. The designer/researcher needs to find out the essentials of the topic: what are the prerequisites, what is it leading to, how can history help in the design of instructional activities, what can we learn from existing research and experiences, what contexts and problem situations can be used.

The didactical phenomenology (Freudenthal, 1983) results in a prototype of instructional materials, or some very first draft activities, that are tried out in the classroom. This is the start of the cyclic process in which the ideas and the materials are constantly revised. From the point of view of research, the observed individual learning processes of both the teacher and the student who participate in the experiment, have considerable impact on the development of a prototype of instruction for the mathematical subject at issue.

mathematization

The first prototype functions as a model of the intended learning process. The prototype is tested in the class; by means of observations, and interviews with students and teachers, evidence is collected on how well 'it works'. The collected data and experiences give reason to reflect on the first trial and to adjust the ideas and the instructional materials. For the students, the learning process is that of progressive mathematization, for the developer/researcher the learning process can be characterized as progressive theorization. Moreover, at the level of the researcher this implies the development of theory on the teaching (and learning) of the mathematical subject at issue. As said earlier, both theory and materials evolve.

Figure 3 shows two versions of the same 'Hats and Glasses' problem. On the left is a composed version of the problem as it was used in the first trial -- the pilot --, and on the right is the revised version as it was used in the next iteration -- the field test. The intention of this problem is to have students develop a reasoning strategy in which they will exchange hats for umbrellas, and make new combinations. The pilot showed that although the pictures may invite for this strategy, the questions don't. Students could reason that a hat was more expensive, but most students answered question 2 using a 'guess & check' strategy. They did not make the connection between questions 1 and 2. Therefore some intermediate questions were inserted in the revision. From observations during the next try-out we learned that students did discover the pattern and used the exchange principle to solve the problem. We learned that some more structure was needed to achieve the intended goal.


Hats and Glasses (pilot version)
$58.00
$52.00
Hats and Glasses (field test version)

1 Which is more expensive, a hat or a pair of glasses?

2 What is the price of one hat?
What is the price of one pair of glasses?
Explain your reasoning.

1 Without calculating the prices for a hat, or a pair of glasses, which is more expensive? How much more expensive?

2 Use the two pictures to make a new combination of glasses and hats, and write down the cost of that combination.

3 Make a combination of only hats or only glasses, and find its price.

4 What is the price of one hat? And of one pair of glasses?

Figure 3. Two composed versions of the 'Hats & Glasses' problem.

interaction

The researcher/developer is not one individual who invents ideas and tries them out. The work is usually done in a team. Before the first version is ready to be tried out, already many drafts have been discussed with colleagues. The discussion is an essential part of the development process; it gives the designer the opportunity to reflect on his ideas. During and after the try-outs there is also a continuous interaction with colleagues, teachers and students. Experiences and findings are shared and open to scrutinize. Developmental research is often a team effort.

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integrated learning strands

In RME and consequently developmental research, teachers, students and researchers are seen as equal partners. One cannot live without the other. All parties involved, learn during the developmental research project. Since it is a team effort, everybody should be open to learn from the other: The learning strands of the different parties are interwoven in service of the prototype that is developed, and the theory about its teaching and learning.

To create an environment in which the persons can learn from each other, one needs to be careful with the language used. To facilitate the communication, understandable and clear language is critical. It is therefore that developmental researchers do not use much of the traditional research jargon and rhetoric.

a comment on the researcher

In the above paragraphs we have used different terms for the person who carries out the developmental research. It is hard to find an appropriate term: researcher, developer, designer. The developmental researcher is a mix of all three. But each of the words carries a meaning that not necessarily means the same as intended. Freudenthal used to call the staff at the institute 'engineers', and maybe that is the best word to describe the developmental researcher.

3 ALGEBRA IN A NEW CURRICULUM

The examples provided so far, come from an instructional unit from the *Mathematics in Context*[*] project. Before we describe in more detail the content and development of this specific unit, we will give a short overview of the vision on school algebra in the MiC project.

The Dutch experiences in research and development in mathematics education of the past decades, the philosophy of RME, and the *NCTM Curriculum and Evaluation Standards for School Mathematics* (1989) form the base for the approach towards algebra in the Mathematics in Context project. In this project, algebra is characterized as:

> "... the study of relationships between variables (the study of joint variation). Students learn how to describe these relationships with a variety of representations, and will be able to connect the representations. The algebra is used to solve problems, and students learn how to use algebra in an appropriate manner. The latter includes making intelligent choices about what algebraic representation (if any) to use in solving a problem. Algebra (its structure and symbols) is not a goal on itself. Algebra is a tool to solve problems. The problems that arise from the real world, are often presented in contexts, they are realistic problems in the way described earlier."

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[*] The *Mathematics in Context* (MiC) project is a NSF-funded project carried out by the University of Wisconsin–Madison and the Freudenthal Institute of Utrecht University in The Netherlands. The purpose of the MiC project is to create a comprehensive mathematics curriculum for grades 5 through 8 that reflects the content and pedagogy suggested by the NCTM *Curriculum and Evaluation Standards*. The materials being developed are based on the beliefs that mathematics should make sense to students and that mathematics is used to make sense of the world around us. The philosophy is based on the theory of Realistic Mathematics Education, and on the belief that mathematics is fallible, changing, and, like any other body of knowledge, the product of human inventiveness. In the MiC project, the content is organized into four combined strands--number, algebra, geometry, and statistics and probability. Over 40 instructional units spread out over the four strands have been developed.

This vision has been worked out in about 13 instructional units. The units in the algebra-strand of the curriculum are strongly interwoven. They are organized around three themes. The theme *Expressions & Formulas* deals with representations of patterns and regularities. Geometric patterns are used to make formulas, and an important aspect is generalizing. The second theme is *Equations*, in which means to represent restrictions and constraints are developed. This theme culminates in optimalization problems and simple linear programming situations. *Graphs* is the name of the third theme. Graphs are used as representations of processes as growth and change.

There are many connections between the units within a theme, and between units from different themes. Topics are being revisited several times during the four years, so students get the opportunity to learn at their own pace. We take ample time to develop the conceptual understanding. A variety of informal and pre-formal methods are used to solve problems, and students are stimulated to use their own strategies to make sense of the problems, and then solving them. Algebraic thinking is more important than algebraic manipulating. Not till late grade 7, and grade 8 the algebra is somewhat formalized. At the end of the middle school we harvest what has been sowed. We do want the students to progress from the informal level to the more sophisticated formal level.

implications for teaching

The approach chosen in the algebra strand -- and also in the other strands of the MiC curriculum -- is that of revisiting the mathematical topics, and slowly but thoroughly developing mathematics. The step towards formal algebraic manipulations is made steadily. At first, the context supports the manipulations. Even when the step towards a higher level of abstraction and formalization is made, the students can always 'fall back' on the context. This approach has implications for instruction. The teacher needs to allow students to develop the mathematical concepts at their own pace. The role of the teacher becomes that of a facilitator and guide in the learning process of the students. She facilitates the students' reinvention of mathematics by means of leading class discussions, knowing how to deal with different answers, and so on. This new role is not easy, but the experiences so far show that more students are learning and enjoying mathematics (Wijers, 1995).

4 AN EXAMPLE: SOLVING SYSTEMS OF EQUATIONS

Research has shown that students have much difficulty with algebra topics as "making equivalent expressions, substituting numbers and variables, and solving systems of linear equations with two or more unknowns" (see Booth 1988, and Wagner & Kieran 1989). Herscovics and Linchevski (1994, in press) identified cognitive gaps related to translating word problems into algebra, translating functions from a tabular representation into graphs and equations, operating with or on unknowns, and operating on a equations as a whole.

As said earlier, much of this research has taken the traditional algebra curriculum as a given. To overcome the difficulties identified to the mathematics of 'solving of equations' we chose for an alternative approach, both in the organization of the content and in the pedagogy. Unlike Kieran (1994) who chose to introduce variables on the functional level, we used quantities, and started with the development of variables as unknowns. From history we can learn that only in the late 1500s, Vieta was the *first* mathematician who used letters for both known and unknown (values) of quantities in equations. As Streefland (1995) indicated, we should also follow the historical development of mathematics when we are designing new ways of teaching and learning mathematics.

Comparing Quantities

Taking the existing research findings, the history of mathematics, common sense, and the professional experience of the developer/researcher into account, a prototype for the instructional unit *Comparing Quantities* that deals with the mathematics of 'solving systems of equations' was designed. *Comparing Quantities* is a sixth grade unit (for students of 10-11 years old) in the algebra strand of MiC. It is the introductory unit of the theme 'equations'.

The design of the unit started with the collection of realistic situations with the potential to naturally provoke strategies as exchanging and making linear combinations. The problems would be represented in pictures and stories to stay close to the world of the students. The teaching-learning situations needed to offer an opportunity for the students to create and develop mathematics and tools to organize and describe their actions. These tools could then be formalized as the mathematics of unknowns, variable, and solving systems of equations. Reasoning had to play an important role as an alternative for the structural approach as often done in the traditional algebra course. A goal was to have students learning to compare, and not studying the algebra of equations. By choosing for a bottom-up approach, the students could create equations themselves, and could develop a conceptual understanding of solving systems of equations.

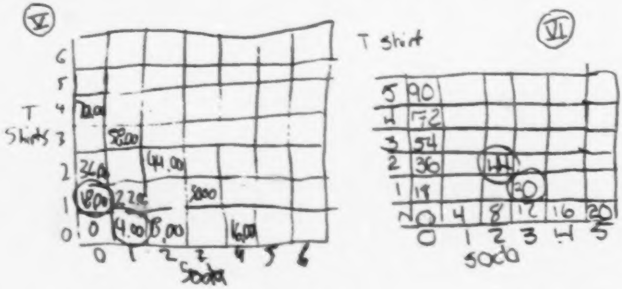

Figure 4. Combination Chart.

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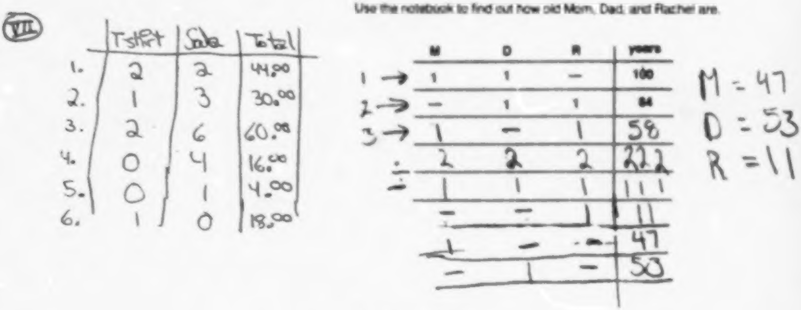
The unit starts with problems that provoke the use of 'guess & check' strategies (see figure 2), varying from random guess and check to more sophisticated 'trial and improve' strategies. A combination chart is introduced as a handy tool to organize and represent the information (figure 4). Moves in the chart represent an exchange of items. Students can analyze patterns in the chart -- e.g., going one up and one left means $14.00 less -- and they can repeat this move to get on the edge of the chart, to create a combination with only one kind of item in it. Besides a handy tool to solve systems of equations, the combination chart involves many other mathematical activities like 'searching for patterns', 'applying tables of multiplication', 'smart calculating'. The reasoning and exchange strategies (see figure 2) from the third category of strategies that are elaborated on in the unit. These strategies are used to discuss patterns and the different ways of notation.



	Tshirt	Soda	Total
1.	2	2	44.00
2.	1	3	30.00
3.	2	6	60.00
4.	0	4	16.00
5.	0	1	4.00
6.	1	0	18.00

Use the notebook to find out how old Mom, Dad, and Rachel are.

	M	D	R	years
1 →	1	1	–	100
2 →	–	1	1	84
3 →	1	–	1	58
	2	2	2	222
÷	1	1	1	111
	–	–	1	11
	1	–	–	47
	–	1	–	53

M = 47
D = 53
R = 11

Figure 5. Notebook.

A fourth category of strategies that is introduced in the unit is the notebook (figure 5). This strategy can be best compared with a notebook used by waiters in a restaurant: Every row is a new order. The notebook is of a more general character than the other strategies. It can be used for any number of items, and it is an efficient way to record new combinations. The notebook can be used to solve the birthday problem as, but the combination chart is not an appropriate strategy. The notebook is a matrix with the coefficients of the items in it. In the unit, students realize that the notebook is a more powerful strategy than guess and check, and that a good reasoning strategy can sometimes be quicker to solve a problem than using the notebook. At the end of the unit, all these strategies are related to each other, and students learn that they are mathematically isomorphic. The next step is to write the information of the problem in equations, and solve the problem using these equations. There are many moments in the unit that give the opportunity to write equations (as in the exchange strategy), but only at the end of the unit, the concepts 'unknown' (and variable) and 'equations' are formalized. In this way students develop an understanding of the equations and the role and meaning of the variables, and they can relate the meaning always to the context of the problem situation. As said before, students are not forced to

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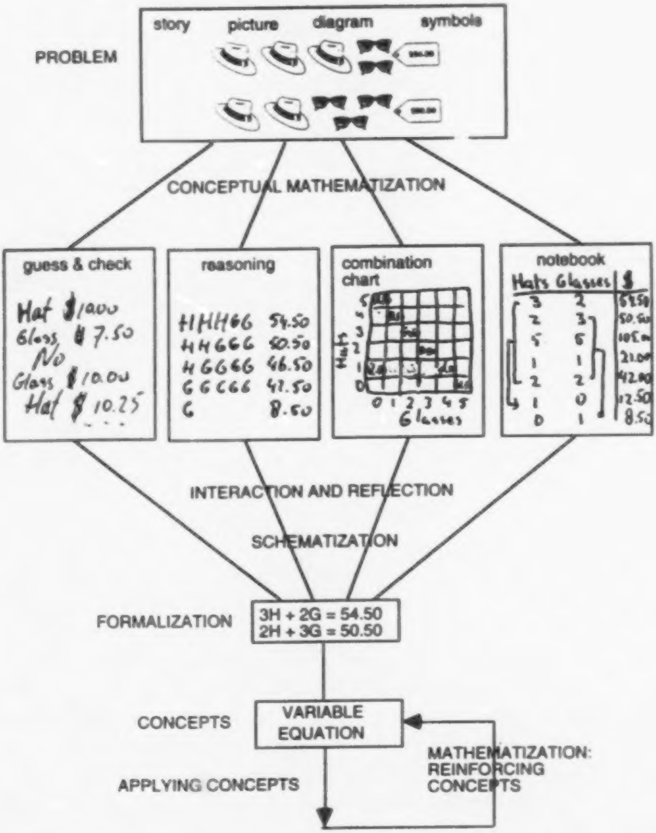
quit using the more concrete pre-formal strategies. The goal of the unit is that students start using variables and equations, and that they realize where they come from. Students can always use any of the strategies mentioned to solve a problem. A strategy that they feel comfortable with, and that is appropriate for the problem situation.


PROBLEM
story
picture
diagram
symbols
CONCEPTUAL MATHEMATIZATION
guess & check
Hat $10.00
Glass $7.50
No
Glass $10.00
Hat $10.25
reasoning
HHHGG 54.50
HHGGG 50.50
HGGGG 46.50
GGGGG 42.50
G 8.50
combination chart
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Glasses
0 1 2 3 4 5
notebook
Hats Glasses $
3 2 54.50
2 3 50.50
5 5 105.00
1 1 21.00
2 2 42.00
1 0 12.50
0 1 8.50
INTERACTION AND REFLECTION
SCHEMATIZATION
FORMALIZATION
3H + 2G = 54.50
2H + 3G = 50.50
CONCEPTS
VARIABLE
EQUATION
APPLYING CONCEPTS
MATHEMATIZATION:
REINFORCING
CONCEPTS

Figure 6. Structure of *Comparing Quantities.*

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We can summarize the structure of the unit with the diagram shown in figure 6. This diagram is a kind of 'map' of the unit. It shows the progression of the development of the mathematical concepts. The problems in the unit are presented in many different ways like pictures, stories, diagrams, symbols; Students get in touch with and learn several informal and pre-formal strategies to solve these problems like guess and check, reasoning (including discovering and using patterns), combination chart, notebook. The strategies and representations are conceptual mathematizations of the problem. By interaction and discussion, students (with the teacher) reflect on these representation, and they are formalized using variables and equations. At the end of the unit, students then apply the concepts and formal representations to solve problems. Realistic situations play an important role in the development of the mathematical concepts. First, they are the world from which mathematics arises; The realistic problems are the source from which students develop the mathematics. Second, the students apply their mathematical knowledge to solve problems in realistic situations.

Figure 6 is the result of several cyclic iterations of research and development. In the following we will shortly describe the process that resulted in this version of the unit.

Development of Comparing Quantities

Very first versions of parts of the unit were tried out in a Dutch classroom to find out if the ideas of the designers were feasible. It appeared that students were more creative and could perform on a higher cognitive level than was anticipated. The step to more sophisticated strategies and models to solve systems of equations was not as hard as we at first thought. Students brought in much more 'knowledge' -- or common sense? – of solving these kind of problems than we anticipated. As long as the problem made sense to the students – that they could realize or imagine what the problem was about -- they made a start to solve it, and in most cases could solve the problem. We learned that students could do more than we might have thought at first. This resulted in the pilot version for the American MiC project.

During the pilot we found out that the unit was too open. Students needed to be challenged more and sometimes they needed more direction to make the step to a higher level of abstraction. From observations of the lessons, and from analyzing student work, we learned that the unit needed a little more structure to help students make the step to a higher level of mathematical sophistication. The examples in figure 3 show what kind of decisions were made in the revision process.

As part of the experiments, an assessment task was designed to use at the end of the three weeks that students had worked on Comparing Quantities. During the pilot, two versions of the task were administered to find out if students had learned the important mathematics in the unit, but also to find out the differences between offering open versus structured problems (van Reeuwijk, 1995b). In the structured problems, a specific strategy was provided and students were asked to use the given strategy. The results showed that students perform better when an open version of the problem is used. In the field test, only the open version of the task was administered. The results confirmed the earlier findings, and it also showed that when students are offered the opportunity to choose a strategy that they think is most appropriate, and that they feel most comfortable with, they perform better, and we can get much more information on students' thinking and performance.

In the process of revising the materials, the 'engineer' has to face a number of dilemma's like: How much structure to put in the materials, how to distinguish between core and optional problems, what to put in the teacher guide, and what to put in the student book, and so on. So far we have learned that teachers are much better in focusing on the essentials of the unit, and on making intelligent choices, the second time they teach the unit.

The design of a unit like *Comparing Quantities* is a cyclic process. As shown above, goals and 'material' influence each other. It is not a linear process in the sense that all the goals are stated beforehand and the content and form are sought to 'fit' these goals. The contexts , the instructional materials, and the outcomes of the experiments can lead to the formulation of new goals and also to the formulation of new theoretical ideas.

5 DISCUSSION AND SOME CONCLUDING REMARKS

We have used *Comparing Quantities* as an example of developmental research, and to illustrate how the philosophy of Realistic Mathematics Education can play out in practice. This little project is only a good start for more research on students' knowledge of early algebra, especially on reasoning and dealing with variables and equations. Streefland's research (1995) is a continuation and elaboration of the ideas developed in the design and testing of *Comparing Quantities*. The results so far show that we are on the right track, and that the approach of using own productions and constructions and moving from informal to formal through a variety of strategies, has much potential as a start for developing the concepts variable and equation. More research is necessary to confirm these preliminary results. Research is also needed on how the development of these mathematical concepts is continued. The shopping problems used in *Comparing Quantities* and in Streefland's research lead to only one interpretation of systems of equations: The letters always stand for the unknown prices. But how to deal with systems of equations where the letters represent the quantities. Then we have to step away form the context of shopping problems, and operate with equations that have a different meaning. Issues like these have to addressed in future developmental research

integration

We have mentioned 'integration of learning strands' as a key idea of RME. Research and development are integrated in developmental research. Much of the traditional research and reform in mathematics education, however, focuses on one aspect of the teaching-learning process There is often a separation of assessment from curriculum development from teacher training and support. As we believe that a separation in single courses algebra, geometry, and calculus is not a very fruitful way to organize mathematics instruction, the different aspects of education should also be integrated in reform projects: Material development should be integrated, with assessment, with instruction, and with teacher education and teacher support. A real reform and innovation of mathematics education can only take place if attention is paid to all these aspects in a coherent way.

closing comments

The cyclic process and the integration of research and development leads to improved versions and to the development of the theory or philosophy underlying he curriculum. The testing of a unit in the class is a very useful and necessary phase in this process. Only then we can really find out what is possible. Making a revision is not an easy process. Information needs to be weighed and choices must be made. In this process the wishes and limitations of the teachers, the students and the designers must all be taken into account. And of course good materials do not guarantee good education, but they help (Wijers 1995).

In this paper developmental research is described in the ideal way. Idealistically the researcher moves through cycles of integrated research and development, of theory and practice. The result is a prototype or demo of materialized ideas for a certain topic of mathematics, that are ready for instruction in the classroom. Developmental research is to be placed somewhere in the middle on the scale of pure theoretical abstract research with no practical applications on one side, and writing textbooks on the other end. Developmental research is certainly more than pure curriculum development, and it is research with strong practical applications, because it is conducted in practice. I am not even sure if it is possible to do pure research in mathematics education. Also the psychology of mathematics education should realize that research in mathematics education is a 'design-science'.

My main and first reason and motivation for the work I am doing on mathematics education is to help developing ways for students and teachers to learn and teach sense making mathematics in a sense making way. This belief often results in a too strong emphasis on the development part of the work. When choices have to be made, teachers and students are number one, and the research community comes second. It is too bad that funding agencies and other policy makers often view developmental research as only curriculum development. This is caused by the traditional view that research and development are separated areas. Because of these traditional views, it is hard to claim and reserve money and time for the research component in curriculum development projects.

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ASPECTS OF THE MATHEMATICS - CULTURE RELATIONSHIP IN MATHEMATICS TEACHING - LEARNING IN COMPULSORY SCHOOL

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to Giovanni Prodi

The purpose of this contribution is to investigate some cognitive and didactic issues regarding the relationship between "mathematics" and "culture" in teaching - learning mathematics in compulsory school. Our attention will focus, firstly, on how everyday culture may be used within school to build up mathematical concepts and skills; secondly, on the contribution that mathematics, as taught at school, may give to everyday culture to allow (and spread) a "scientific" interpretation of natural and social phenomena and, thirdly, on teaching mathematics as a part of the scientific culture which ought to be handed over to the new generations.
We will try to help make clear some potentials and some intrinsic limits of teaching mathematics in "contexts", pointing out the role the teacher has to play to make the best of such potentials and overcome such limits.

1. Introduction

Teaching and learning mathematics in school involve different aspects of the relationship between mathematics and culture such as:

-- the problem concerning how the teacher can use real world situations to build up and/or justify and/or apply mathematical knowledge, and the effects of this usage on out-of-school culture;

-- the relationship between mathematics as taught at school and mathematical experience of students, which is prevalently implicit, in everyday life contexts (for instance, handling money or employing everyday electronic devices);

-- the relationship between mathematics as taught at school and mathematics for mathematicians and other specialists who sistematically make use of advanced mathematical tools.

Our group activity, started in the '70, has dealt with implementing and testing projects to teach mathematics in primary school (grades I-V), in comprehensive school (grades VI-VIII) and, recently, in junior high school (grades IX-X) (see Boero, 1989a; 1989b; 1994b). These projects have shared a characteristic since the beginning: they all sistematically make use of everyday life contexts and real world problem situations to justify, build up and apply mathematical knowledge (cf the first aspect quoted above). Most of our research in mathematics education has focused on the potentials, difficulties and problems involved in such a basic choice, which will be dealt with in this report.

This report will take into consideration:

-- how students' everyday life experience can be used at school to build up concepts and mathematical skills (see § 3);

-- what mathematics as taught at school can give (through students) to out-of-school culture in order

to allow for (and spread) a "scientific" interpretation of natural and social phenomena (see § 4);
-- how to manage the transition to teaching mathematics as a relatively independent part of that scientific culture which ought to be handed over the new generations (see § 5).

Our report will detail the specific difficulties students may meet in relation to these problems, and the different roles the teacher has to play in each of them (see § 7).

The concept of "field of experience" (Boero, 1989a; 1992; 1994b) will be considered in order to consistently frame the different aspects of the mathematics-culture relationship involved at school and to show their functional connections.

2. Theoretical Background

As far as the words "mathematics" and "culture" used in this report are concerned, they will refer not only to the knowledge presently characterizing professional mathematicians and traditionally learned people, respectively, but also:

-- "mathematics" as:

* mathematical topics related to non-mathematical activities and knowledge independently of their level of explicitness, going from the illiterate seasonal laborer's mathematics to the accountant's;
* activities based, depending on those who carry them out and the conditions in which they are carried out, in a more or less explicit and conscious way on elements of the mathematical knowledge. So, for us, mathematics includes not only mathematical concepts and algorithms but also activities such as problem solving, mathematical modeling, production and demonstration of conjectures etc. carried out by anyone.

> We have taken into consideration the latter for two reasons: first, the importance of problem solving and modeling activities for the present work of mathematicians (inside and outside mathematics: from the algebraic treatment of geometric problems to the probabilistic modeling of some biological phenomena); second, the hypothesis (we share) according to which there is a genetic link between activities, processes and conceptualization (for recent studies on the subject refer to Sfard, 1991 and Tall, 1994);

-- "culture" as, according to current anthropological interpretations, any intellectual or material practice shared by social or ethnic groups, which is socially recognizable, communicable and transmittable.

> According to this concept of culture, the so called "material culture" (for instance, agriculture as practiced in different areas all over the world) belongs to "culture" as much as religion or philosophy or mathematics do. But "culture" as considered above is actually made up, through history, of many "cultures": cultures of different countries, cultures of different social or ethnic groups living in the same area, cultures which characterize some institutions (such as school). Everybody's cultural background retains traces of different cultures and everybody, depending on circumstances, appeals, in a more or less conscious and creative way, to those cultures he has experienced.

Regarding the general cognitive and educational issues of our survey, reference will be made to Vygotskij's hypotheses (Vygotskij, 1978; 1990) concerning the relationship between learning and development, the teacher's mediating role and the cognitive functions of semiotic mediation tools.

With reference to Vygotskij (and Leont'ev and Davydov too), it is worth noting that the definition of "culture" we have chosen is consistent with their considering culture as a historical phenomenon rooted in intellectual and material social practices. The importance of "activity" within mathematics is also consistent with the importance they generally attach to "activity" in knowledge forming.

> We may add that the definitions of "mathematics" and "culture" we have chosen might be somehow incompatible with theoretical reference frameworks in the domain of learning pesychology, where knowledge forming is considered a personal process, induced by external stimuli and oriented by social constraints, but basically carried out by inner adaptability mechanisms. In our case, the reference to Vygotskij's hypothesis about the social forming of knowledge within the historical-cultural context seems to make the definitions we have chosen, derived from anthropology, compatible with learning issues.

As far as general educational issues are concerned, we will adapt some concepts developed by the French School of Mathematics Education to our requirements (i.e. the "tool/object dialectics" by R. Douady, 1985 and the "didactical contract" by G. Brousseau, 1984).

> This adjustment may imply incorrectness and inconsistencies with the original theories. Indeed it must be taken into account that mathematics as considered by French theories differentiate from mathematics as considered in this report as well as the theoretical references within the cognitive field which refer to Piaget's constructivism (see Boero, 1994a, for a discussion about the issue).

Regarding the different aspects of the mathematics-culture relationship within teaching-learning mathematics as considered in this report, we will carry out the analysis using, as a way to conceptually unify all issues involved, the concept of "field of experience" (Boero, 1989a, 1992, 1994b), which was introduced in order to analyze the problems met when, in teaching-learning mathematics, contexts the student are acquainted with are referred to. In short, saying "field of experience" we mean a sector of human culture which the teacher and students can recognize and consider as unitary and homogeneous (examples of which are the field of experience of the "sun shadows"and that of "purchases and sales"). Obviously, in the long run, arithmetics too may become a "field of experience" for students. In studying teaching-learning problems related to a given field of experience, the complex relationships which is developed at school between the student's "inner context" (experience, mental representations, procedures concerning the field of experience), the teacher's "inner context" and the "external context" (signs, objects, objective constraints specific of the field of experience) must be considered.

In this report we will consider the evolution of the student's inner context helped by activities organized and guided by the teacher within appropriate "fields of experience". In certain real world fields of experience, he/she may acquire mathematical tools and thinking strategies which he/she will use to think and act more effectively within the same or within other fields of experience. These tools may also become the basic elements to approach (through the teacher's mediation) the mathematical fields of experience.

In this perspective, the problem of the relationship between "culture" and "mathematics teaching-learning" plays a key role if we want to understand the contributions which real world contexts may give to the development of mathematical knowledge and skills, and the contributions that mathematics

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may give to the cultural mastering of different real world contexts. In turn, such understanding seems to be necessary to clarify the potentials, limits and variables governing the effective use of real world contexts in teaching mathematics.

Concerning this issue, we remark that real world contexts are generally used in teaching mathematics in order to connect studying mathematics with out-of-school motivations and applications. Such use often takes on ideological and social connotations which may lead to:
i) undervalue the difficulties sometimes involved when the complexity and difficulty of the subject matter (mathematics) are interlaced with the difficulties regarding the cultural mastering of some real world contexts (which are of course experienced out of school but not explicitly and rationally as the mathematical modeling process requires);
ii) not make the best of the potentials of working in real world contexts as to the development of skills and attitudes needed for activities within the domain of mathematics;
iii) disregard one of the tasks school has to perform that is handing over the new generations mathematics as a science relatively independent of its applications, or disregard (when such task is undertaken) the difference between mathematics considered as a tool (often used in a not fully aware and explicit form) to act in real world contexts and mathematics considered as a science relatively independent of its applications.

As to i) (cultural mastering of real world contexts) this report suggests distinguishing:
-- real world fields of experience which in out-of-school life are already "mathematised" (such as those usually involving measurement of lengths, time and weight or handling money);
-- real world fields of experience in which the mathematical modeling activity carried out at school may be clashing with conceptions rooted in common sense, or anyway it cannot rely on sufficient levels of mathematisation already existing in everyday culture (a good example is the transmission of hereditary characters).

> We think that such distinction is important to make clear how, in the first case, the teacher's work can rely on out-of-school experience to develop concepts and mathematical procedures and to build up higher levels of awareness and explicitness regarding the mathematical tools and processes involved; while, in the second case, this does not happen and sometimes the teacher has to work against conceptions opposed to mathematical modeling.

As to ii) (potentials of activities related to real world contexts) this report will try to point out how some basic skills and processes involved in mathematical activities (such as linguistic-reasoning skills, meta-cognitive processes etc.) can be developed by implementing the potentials of working in real world experience fields.

As to iii) (relationship between mathematics as a tool to act in real world contexts and mathematics as an independent science), this report suggests that, on one hand, there is no gap between some skills and some concepts which can be built up by working in real world fields of experience and used for working in mathematics but, on the other hand, there are gaps between everyday thinking and thinking through mathematics as well as between everyday thinking and mathematical thinking. We believe that teaching mathematics must be concerned about all this ("continuity" and "discontinuity").

3. Everyday mathematical experience and school mathematics

Everyday culture includes social practices (i.e. money-goods exchange, measurement of common physical magnitudes such as lengths, weights etc.) where fundamental mathematical concepts, properties and strategies are used (sometimes as implicit operating tools). There are also objects (referring to the preceding examples: money, rulers etc.) which imply, in order to be used according to social conventions, substantial mathematical knowledge and skills (Boero, Carlucci, Chiappini, Ferrero & Lemut, 1994).

Referring to fields of experience such as "purchases and sales" or "calendar" the teacher may introduce out-of-school social practices (for instance, money-goods exchange) into the class and use objects and language expressions which enhance the resonance of school activity with the students' out-of-school experiences. By accurately selecting the problem situations to submit, the teacher may stimulate students to nourish (within the short-lived work at school) their cultural development and overcome naive ideas. These are processes usually occuring out of school in special environments (such as those studied by Carraher, 1988) and used to occur, at large, in the ages preceding the widespread development of schooling institutions in several societies (see Bishop, 1988).

> As to children's "naive" ideas, some children, on entering primary school, may show such ideas, for instance, in reference to the buying power of money or purchase procedures. But we have seen that for the child to achieve an "adult" view rather quickly it is enough to refer to external rules and constraints (generally the process is helped by what the child meanwhile experiences out of school).

3.1. Our studies show that when systematic and (where possible) student-involving and realistic activities are performed at school, not only is it possible to build up skills to solve simple practical problems, but also to help "theorems in action" (Vergnaud, 1990) to emerge, such as the distributivity of multiplication with respect to addition when working out the cost of 3 items which cost 420 liras each (Boero, 1988, 1992) and the additivity of length measures in solving the problem of using a ruler 20 cm long to get right the heigth of a plant longer than the ruler (Boero, 1994a).

It is also possible to help the development of significant cognitive processes: anticipation (Boero & Shapiro, 1992), hypothetical reasoning and working out different types of hypotheses and strategies (Ferrari, 1989; 1990; 1992; Boero, 1990; Boero, Ferrari & Ferrero, 1989; Boero, Ferrari, Ferrero & Shapiro, 1994).

3.2. Research problems.

A considerable problem involved in teaching "realistic mathematics" (Treffers & Goffree, 1985; Treffers, 1987) or teaching mathematics according to the "situated cognition" perspective (Rogoff & Lave, 1984; Lave, 1988; cf Vanderbilt Group, 1990) concerns how systematically and continuosly to perform activities in a given everyday life field of experience.

There are different options, in particular:
a) different everyday life fields of experience are episodically recalled while submitting word problems (if necessary submitted as "story problems");
b) an everyday life field of experience is worked away for a long time and the work is basically directed by the requirements of building up mathematical concepts and skills;
c) an everyday life field of experience is worked away for a long time and the work is basically directed by the requirements of the development of the knowledge concerning the field of experience itself (in this case the concern about developing activities rich in mathematical meanings and implications is not solved by chosing suitable problem situations but by choosing a field of experience rich in mathematical potentials).

Certainly even an isolated "story problem" which the teacher manages to emotionally involve the students in stimulates them to start context-dependent strategies (Lesh, 1985), but in our opinion it is necessary to distinguish between:
-- recalling (through the text of problems) of context-dependent strategies;
-- building up and development of solving strategies and mathematical concepts and skills through class activities related to the everyday fields of experience.
Referring to such distinction, our research has shown that the potentials of using everyday life fields of experience in mathematics teaching according to a) and b) are very limited as far as the development of mathematical skills is concerned, while the approach c) is much richer in potentials (see Boero, 1988; 1992; 1994b; Boero, Ferrari, Ferrero & Shapiro, 1994; see Vanderbilt Group, 1990, for similar conclusions concerning general education).

Now, let us consider the skills built up within an everyday life field of experience: they must be made recognizable and usable for a more systematic mastering of the field of experience one is working in and in order to reinvest them in other (mathematical or non-mathematical) fields of experience. The teacher must therefore guide the process of making explicit (through standard language forms and, as far as it is actually necessary, through mathematical language forms) the mathematical knowledge built-in in the activities performed at school in the everyday life experience fields. This transition from implicit operative tools to explicit mathematical objects shows some difficulties.
In our opinion the main research problem lies in identifying and correctly managing the differences between school mathematical knowledge and mathematical knowledge attached to social practices (i.e. everyday economic or technical activities etc.). The issue has been clearly put forward by Carraher, 1988, as far as the relationship between problem solving strategies performed by children in the street and strategies as taught at school is concerned.

Literature shows different approaches to this problem. Some general approaches (like the one proposed by Vygotskij, 1990, chapter VI) are related to the different nature of "everyday concepts" and "scientific concepts", in particular as far as explicitness and systematicity of knowledge they refer to are concerned. Other approaches, expressively concerning primary mathematical skills, refer to the fact that "street mathematics" is basically mental and oral, while mathematics at school is basically written (see Carraher, T., Carraher, D.W. & Schliemann, A., 1987). Other researchers take into consideration the fact that the problem met out of school has no educational connotations, while the problem met at school is loaded, by the teacher submitting it, with educational purposes (Brousseau, 1986). In addition the relationship between teacher and student ("didactic contract") greatly affects the solving strategy, the selection of the main items of the solution to stick to mind etc.

In our opinion the clash between school mathematics knowledge and out-of-school everyday mathematical experience should be considered in connection with the different levels and domains in which it occurs. If, on one hand, mental calculation strategies used in subtraction are different from those used in standard written calculation, on the other, this difference has not the same nature of the one concerning, in geometry, the relationship between mathematical proof and empirical verification of a statement as well as the one concerning, again in geometry, empirical "commensurability" and mathematical "incommensurability" of the diagonal and the side of a square (see § 5).

Another problem concerning activities in everyday life contexts in order to build up mathematical skills is the possibility for students to live "realistically" the problem situations related to out-of-school fields of experience we are referring to (see Sierpinska, 1994; Verstappen, 1994). Still considering that particular everyday life problem situations may greatly involve junior high school students, it seems much easier to involve primary school students. We also think it must be taken into account that some mathematical fields of experience (arithmetic, elementary probability etc.) may become familiar to junior high school students and suitable for challenging mathematical activities.

4. Mathematics and "scientific" conception of natural and social phenomena

We are interested in the role played by mathematics in giving scientific interpretations of natural and social phenomena which make up the world view gradually drawn up by modern culture and which should be handed over to the new generations through school. With this in mind, the teacher may propose mathematical modeling of phenomena (for instance, sun shadows or transmission of hereditary characters) which are remarkable for the history of culture. Unlike what is stated in subsection 3, in this case school learning cannot be replaced by out-of-school experience, not even at an operating-implicit level, as far as using mathematical tools to solve problems is concerned. Moreover very often the teacher must oppose the student's conceptions worked out within his/her own environment or personal life story. So school should be committed to hand over the new generations a particular view of social and natural phenomena which may be in conflict with other views existing in our society or in other ones.

> The legitimacy of this task is not unquestionable: the mathematical modeling of phenomena enhances a particular way of looking at the world (based on the study of quantitative relations between measurable magnitudes) and tends to emphasize some aspects of phenomena while neglecting other ones (those non-mathematisable or not yet mathematised). So it is not a neutral choice with respect to past and present value systems and cultures. It is a choice which brings out a certain number of problems if one believes that there is no "higher truth" expressing itself in a higher level of mathematisation but rather there are different "truths" built up and expressed by different cultural tools. It is worth reminding that within the field of experimental sciences itself the recent debate (dealing with complex phenomena and their non-reducibility to a sum of simple components etc. - see Prigogyne, Thom etc.) wonder about the scientific legitimacy of totalizing cultural operations which consider the mathematisation of the relations between measurable variables as the model of scientific truth.
> However in our opinion the problem of respecting cultural differences for culturally bringing up the new generations may be solved without giving up handing over scientific culture nor giving up significant mathematical modeling of natural and social phenomena but enhancing the training in the historical, philosophic and anthropological fields together with more room, in a comprehensive syllabus, for mathematical modeling (in order to give the necessary knowledge

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of a method - or, if possible - more methods for using mathematics to get to know the world). As far as we have seen in the class, it is possibile for 10/12-year-old children, within their mathematical and scientific training, to be introduced to historical and epistemological elements concerning their activity (see Sibilla,1989, Boero&Garuti,1994;see also Bartolini Bussi, 1994).

4.1. That said, our research (Boero, 1989b; Garuti & Boero, 1992; Garuti & Boero, 1994; Scali, 1994) highlights:

-- the importance of general "principles" and specific "phenomenon conceptions", among the resources students draw from and as part of their way of thinking; it has been noted that conceptions and principles sometimes act as a back-up for the mathematical modeling of phenomena while, at other times, they hinder such modeling, and at other times again, they play an ambiguos role by suggesting some connections between conceptions and mathematical models to teacher and students while, at the same time, strengthening non-"scientific" conceptions of the phenomenon concerned.

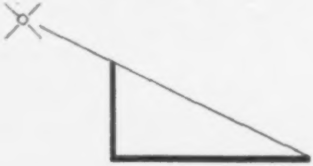
The two fields of experience we have more deeply analyzed in reference to the issue above are sun shadows (for students from 8 to 13) and the transmission of hereditary characters (for 12/13- year-old students).

As to sun shadows, we built up situations which allowed different conceptions to come out both in students already trained in geometric modeling activities and student not yet trained in that. We so classified various conceptions: shadow as an effect of the strength of the sun, as a "double", as an appendix, as an intersection between shade and a surface,... The per cent distribution of these conceptions and how they show (more or less intertwined with one another or "dressed in geometric clothes") greatly depend on the activities previously carried out at school within the field of experience of the sun- shadows. Changes depending on age do not show so relevant.

As to the transmission of hereditary characters, some constants found at the beginning of the activity (and found again after performing probabilistic modeling activities) concerns the mixing of characters taken as the idea about how the phenomenon happens *("children's characteristics are something in between their parents', if children take something after their fathers they take something after their mothers too, and so in short a child is something in between the father and the mother")*, and fatalism or sin as principles to explain why and how unfavorable characteristics are transmitted (in particular, as far as hereditary deseases are concerned).

-- the importance of the sign systems proposed by the teacher in order to stimulate the transition to a scientific conception of phenomena.

Refer, for instance, to the straight lines which represent light beams when passing from students' early conceptions to the mathematical modeling of shadows.



When students are 9/10-years-old or even 11/12-years-old this geometric model (we will call it "shadow diagram"), properly introduced by the teacher through a lively discussion about the shadow phenomenon or enhanced if some student put it forward, greatly affects the idea of *"the higher and stronger the sun, the longer the shadow"* (otherwise it might frequently reappear

later on even after having observed the phenomenon again and again).
The "shadow diagram" seems to modify even the way of thinking about the relationship between height of the sun and length of cast shadows: from observing that *"the sun is low on the horizon and the shadows are long"* students start stating (after having known the "shadow diagram") something like *"if the sun is low on the horizon the shadows are long"* or *"the shadows are long because the sun is low on the horizon"*.

4.2. The following research problems are connected to the experimental data stated above:

-- what are the nature and origin of the conceptions and principles shown by students? In some instances, they seem to be rooted in out-of-school culture (a fatalistic view of hereditary deseases) while, in other instances, they seem to correspond to individual developmental steps of one's way of thinking (shadow length dependent on the sun strength); in other instances again, they seem to correspond to general principles inherent in our intellectual relationship with the world (principle of continuity or symmetry);

- does the mathematics teacher have to refer to students' culture? And if so, how?

The issue is crucial especially when students' principles and conceptions intervene in class activities as cultural obstacles; as we have seen at the beginning of this subsection, this is a crucial issue not only from a cognitive and educational standpoint but also from a cultural standpoint (respect of cultural differences, relations with views of the world which do not adjust themselves to the "mathematical modeling" principle taken as a main road toward knowing and understanding phenomena).

A few comments about the preceding two issues.
From a cognitive standpoint, students' principles and conceptions have been deeply investigated by psychology researchers and different hypotheses have been made in the last hundred years. We would like to recall Piaget's studies on children's mental representations and Vygotskij's critical analysis of Piaget's theory about the characteristics of child thinking, and Vygotskij's hypothesis according to which a continuos interlacing between "everyday concepts" and "scientific concepts" would be happening since school starts teaching science.

In our investigating students' ideas about the phenomenon of sun shadows within an age range from 8 to 12, we think we have met conceptions which, for intrinsic reasons, match geometric modeling well and gradually develop with it (for instance, refer to the idea of shadow-appendix spreading on the surface on which it can be seen). We have also met conceptions (more or less in the same proportion both for 8-year-old and 12-year-old students) which go back (even though they are still in the background and are sometimes connected with more general "principles") such as length of shadow dependent on the strength of the sun. Generally, the conceptions which go back end up clashing both with experimental data and geometric modeling.

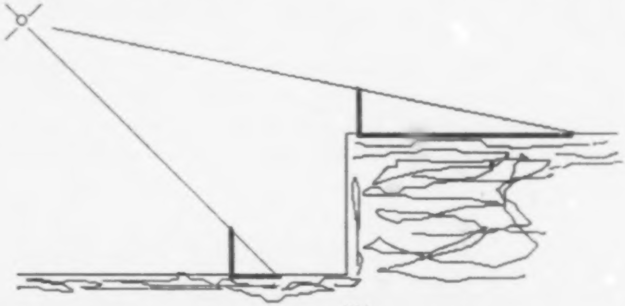
As to the transmission of hereditary characters, the situation seems even more complex: principles and deeply rooted conceptions break through again after some time and variously intertwine with conceptions learned, sometimes showing a need for a connection with everyday experience (according to which hereditary characters depend on several genes and so what happens more often is just the statistical mixing!), while at other times the need is for an explanation of facts which catch students' emotional sphere (hereditary deseases).

When having to cope with such behaviours, it is very hard to make educational choices both because they involve the cultural training in the whole and because it is not clear what long-term aftereffects a scientific teaching which pushes students to forget about (or radically change) their conceptions and principles can have. In some instances, we have had the feeling that working on students' conceptions and principles carelessly may undermine the roots of their thinking and reduce the development of their competence in conceiving and reasoning.

-- situations and ways the teacher may choose to mediate mathematical sign systems, without building up stereotypes (cf. Ferrari, 1992) which replace phenomena.

Actually, if it is true that mathematical signs help mastering the phenomena studied, it is also true that sometimes reasoning on signs replace reasoning on the phenomenon.

Let us consider the "shadow diagram" as an example.
As seen above, this tool is very useful both to understand the connections between height of the sun and length of the cast shadow and to investigate the connections between height of objects and length of cast shadows (making an interesting "bridge" with Thales' theorem: see Garuti & Boero, 1992; Boero & Garuti, 1994).
But when it is time to decide if, at the same hour, the shadow which two vertical nails of the same length cast on two horizontal planes looks the same in the school yard and on the roof of the school, almost 50% students (grade V, with a solid background in studying shadows) answer that the shadow on the roof is longer, giving geometric motivations of the same type as shown below (see Scali, 1994).



5. Mathematics as a specialized and explicit cultural experience

This subsection will deal with mathematics as part of today's scientific culture for its being a specialized activity of mathematicians (particularly of "pure mathematicians") and a component of the basic cultural grounding of intellectual classes in modern societies. So mathematics may provide traditionally learned people with some reasoning patterns ("mathematical rigour") and give some ideas which frame the meditation on intellectual experiences ("infinity").

Unlike § 3 and 4, we will not consider mathematics as a tool to know and work on non-mathematical phenomena and problems but rather its nature of (relatively) independent science and its today's characteristics. Within such framework, we may consider both mathematics as taught at high schools in most countries according to educational aims which are not directly related to applications, and mathematics as a cultural developmental phenomenon (a field of investigation and discovery, variously systematized).

In order for a mathematical field of experience to be developed in the class, the teacher must introduce elements drawn from a scientific culture (different from everyday culture) for all the aspects qualifying it in terms of cultural specialization (sometimes causing epistemological "cuts" with respect to everyday culture: see Balacheff, 1988). Mathematical statements have a specific structure, mathematical proof is a special way of reasoning, some mathematical concepts (such as the concept of infinity or the concept of irrational number) are far from the operating experience of mathematical tools. How can compulsory school students begin grasping these specific aspects of mathematics?

5.1. In our projects, from grade VI to grade X the teacher previously builds up, within the real world fields of experience, the bases for conceptualization and reasoning within the mathematical fields of experience (cf § 3, 4 and 6). Then he creates and manages teaching situations where students can carry out constructive activities (conjectures, demonstrations, reflections...) concerning mathematical "objects" as such (which anyway students already know well as "tools")(see Boero & Garuti, 1994; Garuti & al., 1995). Our studies (in particular regarding the first approaches to mathematics as an independent science) show that during such activities:

-- referring explicitly to problem formulations or to breakthroughs ascribed to historical notables facilitates the students' distancing from their intellectual work and the general and synthetic formulation of its results (Boero & Garuti, 1994; also see Bartolini Bussi, 1994).

> The following experience, carried out with grade VII students, seems worthy: after some hard work on problems of proportionality concerning geometric modeling of shadows and anthropometric regularities,the teacher introduces the anedocte about the problem of getting right the height of the Pyramid as solved by Thales. Then he asks students to identify themselves with Thales and draw up a will listing discoveries made to be handed down to posterity. This situation leads students to produce texts which show two very interesting characteristics:
> • restoring and synthesis of the work done by students;
> • general, and sometimes "abstract" and "conditional", formulation of statements similar to the different formulations of Thales' theorem found in textbooks;

-- comparing the student-produced texts with standard mathematical texts (for instance, those found in textbooks) pushes many students to reformulate their texts or the reference texts in order to have them somehow resembling to one another (Boero & Garuti, 1994).

> Actually, still referring to the preceding example, we saw that when (after drawing up the "wills") different statements of the Thales' theorem were shown in the class, a number of students tried to lead back their texts to the reference text (chosing the "official" text actually closer to their own), while others tried to change the official statement, which was recognized as the closest to their own, in order to demonstrate its affinity with theirs.

5.2. Research problems:

-- how far can the teacher go in pushing the student's constructive activity (personal and social) and when should he/she start proposing cultural models unfamiliar to the class?

The answer is not easy; it seems to depend on many elements, among which: student's age and cultural experience, didactical contract agreed upon in the class, sectors of mathematics and, above all, kind of performances required;

-- what are the potentials and the cognitive mechanisms involved in directly using historical sources (notables, texts...) both when students work mathematically and when they compare their products with "official" cultural models?

We think that, on the basis of ours and other researchers' experiences with very young students (see Boero & Garuti, 1994; Bartolini Bussi, 1994; Grugnetti, 1989), using historical sources may be effective for different reasons: time distancing which helps cultural distancing; comparison and reflection on mathematical topics historical sources are related to; emotional mechanisms connected with one's identity quest started by tasks reminding "the origins"; availability of problem formulations and their solutions expressed through a language which is different from the present one (different formalisms or more direct lexical connections with everyday experience etc.);

-- how can the cultural ground be made ready (and how suitable situations can be created and managed in it) to allow epistemological obstacles to come out? Students' taking into consideration these obstacles seems to be a necessary step for the development of their mathematical culture and their competence in doing mathematics (see Fishbein, Jehiam & Cohen, 1994);

-- what are the pre-requirements needed for a students' productive approach to the fields of experience of mathematics and what are the means to put them together?

As to specific pre-requirements concerning mathematical concepts and skills, it seems to be enough to enable students to master them by dealing with problem situations well rooted in real world fields of experience. However levels of logical consequentiality, generalization and reflection higher than those many students usually have seem to be necessary. In this respect we have seen that 10/11-year-old students already show remarkable potentials concerning reflection and logical reasoning skills needed for working in the mathematical fields of experience. Usually these potentials may be more or less developed depending on students' social and cultural up-bringing.

Our experiences suggest that activities referring to real world fields of experience, conveniently managed by the teacher, may help most students to develop linguistic and reflective skills. In particular, we think that, referring to the situations discussed in § 4, the problems of conflict and evolution of students' conceptions and principles may greatly favour reflective thinking.

6. Connections between the Various Aspects of the Mathematics - Culture Relationship

First of all, it is to be underlined that, in developmental and functional terms, the three types of relationships between mathematics and culture in compulsory school described at § 3, 4 and 5, can be connected, in some cases even within the same field of experience. For instance, referring to § 4 and 5, in the field of experience of "sun shadows", many students can overcome or change their early non-geometrical conceptions of the shadow phenomenon through geometric schematization. Such schematization may then be used to tackle "real" problems within the same field of experience (such as determining heights that are inaccessible by direct measuring; see Garuti & Boero, 1992) and, in a theoretical sense, in building up the field of experience of "rational geometry" (Boero & Garuti, 1994).

Through appropriate educational planning, the work within real world fields of experience may so supply concepts (usually in form of "tools") and skills required to work in the mathematical fields of experience. On the other hand, the activities in these fields of experience are based on forms of reasoning which appear to be rooted in non-mathematical experience and in mathematical modeling experience. In particular, Boero, Ferrari, Ferrero & Shapiro (1994) show how, between grade V and VIII, some out-of-school fields of experience offer resources to develop an initial class "hypothesis game" which gradually enables students to culturally master the field (according to the interpretative

models typical of different sciences), while allowing them to reach a more complex and demanding hypothesis working out (conjectures, interpretations, justifications, ...) in the same or in other fields of experience (including the mathematical fields of experience).
Referring to the external context, the student's inner context and the teacher's inner context (see § 2), working in a real world field of experience allows the teacher to rely on out-of-school constraints and resources of the external context and on cognitive strategies and ways of thinking of the student's inner context (connected with out-of-school experience) to introduce (or develop) signs, procedures and mathematical concepts suitable for solving problems which draw their meaning and legitimation from out-of-school experience. Little by little the teacher (implementing different class activities such as explanation, general discussion, comparison of students' works) gives the first elements (signs) of the external context of a mathematical field of experience (for instance, geometry) and helps the students to build up the first elements of their inner context. To do this, the student may use the signs the teacher has introduced during problem solving activities related to real world fields of experience and may work on his/her reasoning and reflecting on concepts and mathematical procedures used, in those activities, as "tools" (which must become "objects" for activities within the mathematical "field of experience": transition from an extra-mathematical setting to a mathematical setting - see Douady, 1985).
The process of building up and identifying a mathematical field of experience may be furtherly helped by historical sources, official texts etc.

A remarkable aspect of activities referring to real world fields of experience concerns the possibility of developing processes of social construction of knowledge in the classroom, because students' inner contexts and the teacher's inner context may enter immediate resonance on topics referring to common experience. All this may also enhance a favourable climate for productive discussion about mathematical strategies and objects involved in those activities, preparing the ground for discussions in the mathematical fields of experience (see Ferrari & Bondesan, 1991; Garuti & al., 1995).

Still referring to the transition from real world to mathematical fields of experience, it is worth mentioning an interesting research topic. In the framework of activities developed in real world experience fields, we have often seen the conversion of "processes" into "mathematical objects" (cf Sfard, 1991). We think that a thorough study of this phenomenon should be made since the natural transition from activities within real world contexts to the objects the mathematical work is concerned with may depend on it.

Another research topic related to the connections between the three types of mathematics-culture relationships considered at subsections 3, 4 and 5, concerns the study of the cognitive working of the traditional mathematical teaching which is the most widespread in the world, especially at advanced school levels, despite its being hardly appreciated by mathematics education researchers! In it the mathematical fields of experience are built in the class through the teacher explaining definitions, rules and theorems and students being drilled (according to models taught by the teacher). The relationship with real world fields of experience is only achieved through some applications of the acquired mathematical tools.

It is a kind of teaching which works well with some students: it is enough to consider that most intellectual classes of our society, including the specialists who have helped sciences to reach the levels we know today, have been trained through it in the last centuries. On the other hand, its limits are well known as well: it is greatly selective and supplies knowledge which turns out inert for most students (who are unable to use it to solve non-standard problems, especially in real world problem situations).

A research on the cognitive working of the "traditional" way of teaching mathematics might be very useful because:

-- it might help understanding how the student may give meanings to a sign, a definition, a theory introduced or explained by the teacher, and under what conditions this may happen;

-- it might help making clear potentials and limits of the constructivist foundations of mathematics teaching which are very successful among mathematics educators today;

-- it might help the teacher to provide a common ground to educational activities which see students work at building up basic mathematical concepts and skills, and educational activities carried out to have students to get hold of concepts and methods explained by the teacher.

In our opinion, some classic theoretical frameworks of learning psychology may give these investigations the necessary basic tools. We are particularly referring to Vygotskij's analysis of the relationship between teaching-learning scientific concepts and the student' everyday concepts (in particular with reference to what cognitive use of tools and methods systematically and explicitly taught by the teacher the student can make); and also to Ausubel's studies on "meaningful learning" ("through reception" and "through discovery").

7. Other comments on the teacher's role in the class: the didactical contract

In the preceding subsections we have pointed out the different roles the teacher has to play according to the different relations between mathematics and culture which are built up in the instances we have considered:

-- within some fields of experience such as that related to basic trade exchanges (§ 3), the teacher plays the role of a "mediating supporter" of conventions and practices already known by the student at least at a very early stage (or included in the cultural environment he/she comes from);

-- within other fields of experience such as that related to the transmission of hereditary characters (§ 4), the teacher must necessarily play the role of a committed "dissenter" opposing the naive or non-"scientific" ways of thinking of the students and, often, of the same environment they come from. A dissenter who holds a dialogue with such ways of thinking because he/she cannot afford ignoring them and because some embryonic elements able to help the transition to a "scientific" conception of the phenomenon concerned can be found in some of them;

-- in the mathematical fields of experience (§ 5), the teacher must play the role of a "witness of mathematical culture" who has to give students elements (not included in the class experience) able to allow them to compare their mathematical outcomes,and tools for representing such outcomes in order to make them develop and draw together towards official mathematical culture (Garuti & al, 1995).

In our experience of planning and testing educational projects for compulsory school mathematics (for 6/16-year-old students) the complexity of the teacher's role is one of crucial issue; it requires the teacher to be fully aware of the epistemological, cognitive and educational implications of his/her choices concerning the mathematics/culture relationship. Epistemological and cognitive issues have been already considered in the preceding subsections. As to educational aspects, we would like to underline that the difficulties concerning the profession of teacher match those concerning the students when they practice the "profession of students", particularly when they have to comply with the didactical contract and its changing according to the different role played by the teacher.

We think this issue should be examined carefully. We feel that a thorough study of what level of awareness students of different age can have about what they are required to achieve should be made. And also how to achieve such awareness should be investigated. We have the feeling (and the hope!) that most 11/12-year-old students, in good class situations where the pre-requirements concerning reasoning and reflection are complied with, could reach the first levels of awareness as to the characteristics of a correct reasoning regarding modelization of reality (consistency with facts and experimental data, internal logical consistency), as to the limits of a mathematical model compared with the modelized reality, as to the distinction between empirical truth and mathematical reality in specific instances (such as the incommensurability between side and diagonal of a square).

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WORKING GROUPS

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WORKING GROUP ON

CULTURAL ASPECTS IN THE LEARNING OF MATHEMATICS

The group is concerned with various research and debates arising in the community of math education about relationships between CULTURE and mathematical knowledge. Psychological and anthropological approaches of CULTURE are considered.

In the previous PME sessions, attempts were made in order to clarify research directions which were recognized consistent with our interests:

- Informal education and formal mathematical knowledge
- Cognitive processes in learning mathematics and cultural environment

According to PME XVIII session, the members have been invited to contribute to a booklet: previous research, research in progress or review of the literature.

These draft reports will constitute a basis for our discussions at PME XIX and a first step to the preparation of PME XX in Spain. The three sessions will be introduced by contributors of various countries on the following issues:

Session 1

1. Ethnomathematical content related to:

- school math curriculum: materials suitable for classroom use in order to facilitate access to formal mathematics
- alternative project: learning process in classroom and in professional environment, math classroom learning built on students' everyday experiences and backgrounds.

2. Applied mathematics as a source or as a goal in the school curriculum: a case of study with historical aspects.

Session 2

1. Way of using mathematical concepts in a bilingual or bicultural case of study through:

- language in a bilingual situation, using social, linguistic and material resources.
- graphic representation of space: pluridisciplinary approach in a bi-cultural study.

2. Relational aspects in a pluricultural situation with reference to the home and school context and mathematical learning.

Session 3

Relations between different ways to use the word CULTURE and the different purposes according to which it is used in and outside maths education.

What are the fundamental and useful questions to maths education itself and math education as part of holistic education?

Contributors: Marcelo de Carvalho Borba, Marta Civil, Bernadette Denys,
Ceri Morgan, Inés Mª Gómez-Chacón, Paul Laridon,
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RESEARCH ON THE PSYCHOLOGY OF MATHEMATICS TEACHER DEVELOPMENT

Between 1986 and 1989, a Discussion Group on *Research on the Psychology of Mathematics Teacher Development* met at annual PME conferences. In 1990, this Discussion Group was accepted as a Working Group and has continued to meet in this format at PME conferences since then. At PME 19 we hope to build on the foundation of shared understandings that have developed over the past ten years.

Aims of the Working Group

The Working Groups aims to:

- develop, communicate and examine paradigms and frameworks for research in the psychology of mathematics teacher development;
- collect, develop, discuss and critique tools and methodologies for conducting naturalistic and intervention research concerning the development of mathematics teachers' knowledge, beliefs, actions and reflections;
- implement collaborative research projects;
- foster and develop communication between participants;
- produce a joint publication on research frameworks and methodological issues.

Plans for Working Group Activities at PME in 1995

A strong feature of the Working Group for Research on the Psychology of Mathematics Teacher Development has been its cohesiveness, and its wide representation across many countries.

Group members have expressed the need to have a deeper understanding of a range of research methodologies which are particularly appropriate for research in the area of mathematics teacher development. In 1994, the Working Group decided to focus its efforts on the publication of a series of papers as a book with the working title *Research on Mathematics Teacher Development: An International Perspective.*

The preparation of this book provides a unique opportunity to bring together the research expertise of mathematics educators currently working in the area of mathematics teacher development in a wide range of cultural contexts, and to disseminate both their approaches to research and their findings more widely. The book will be dedicated to the collection, development, discussion and critiquing of paradigms and frameworks for research in the psychology of mathematics teacher development.

The Working Group sessions in 1995 will concentrate on the presentation and discussion of the papers to be included in the book.

Nerida F. Ellerton, Convenor

Working Group

Algebraic Processes and Structures

Organisers: Romulo Campos Lins & Teresa Rojano

Since 1990 this Working Group has put a considerable effort into bringing the available perspectives on algebraic thinking and on the teaching of algebra into a more or less coherent whole, or at least making it possible that the various approaches can 'talk' with each other where a convergence seems unlikely. The collections of short papers produced for the annual PME meetings provide good evidence that the group has succeeded to a considerable extent.

The 1995 meeting will mark the closing of a cycle, as the Group prepares to publish a book, in which the results of those years of discussion will appear within chapters written by several of the people involved with the WG's activities.

We think that the 1995 meeting has to fulfill a twofold condition. First, we should reflect on the results reached so far by the WG, particularly aiming at producing for the community a clear picture of what the WG has produced as a consequence of its existence. Second, we should make room for the opening of a fresh perspective for the possible continuation of the group.

In order to achieve those objectives, the following structure is proposed for the 1995 meeting of the *Algebraic Structure and Processes Working Group*:

(i) a brief presentation of the themes discussed in the group since 1990;

(ii) a discussion of the structure of the book, including a reflection about the changes in trends which have happened since the group has started to meet, and,

(iii) the presentation, followed by discussion, of a video-taped classroom episode, centred on a activity involving 'algebra' or 'algebraic thinking'.

Item (iii) is particularly intended at providing an environment in which old and new members of the group will 'work from scratch,' ie, it is intended to 'loosen up' the ties between the work so far produced and what might come to be the WG's future direction. We think this is an important step, as the WG has always produced a strong sense of continuity in its work, something which has certainly proved fruitful, but might as well become an obstacle now that a sort of apex has been reached.

A booklet composed of one-page abstracts of the chapters will be made available to the participants.

WORKING GROUP
GEOMETRY

(Maria Alessandra Mariotti)

Different external representations in the geometrical field: their dialectic relationship with geometrical knowledge.

It seems nearly impossible to conceive geometry without 'figures', drawings , constructions, any kind of images The ambiguity of the term figure has often been pointed out by many authors, the term traditionally refers both to the mathematical object and to its 'concrete' representation, for instance a drawing in a textbook At the same time this ambiguity focuses the deep link between the two aspects, and witnesses the interrelation between images and geometrical ideas

History tells us about the basic contribution given by the reflections on perspective drawing in the development of 'projective geometry' The geometrical theorization of Desargues moved from experiences and theories of outstanding artists and artisans

Recent research dealt in several different ways with the relationship between technical drawing and geometrical education

New software are now available, which provide images differently linked to geometrical topics, these new means open new perspectives on geometrical education

A common point of view from which all these different aspects can be approached is the analysis of the interaction between external representations and geometrical knowledge Particularly, the main issue is how to use the dialectic relationship between external representations and geometrical knowledge in the educational field

Certainly, the presence of new technologies and the availability of purposeful software raise the problem of images anew, but, using new technologies in geometrical education does not exhaust the complexity of the problem At the same time, although their presence adds new elements to the analysis, focusing to computers risks to hide the rich contribute coming from other sources

These and other issues that you will suggest will be discussed

It would be interesting to hear about the current research on this topic, thus those who are directly involved in this field of research are invited to participate and contribute

Whoever interest in contributing to the discussion with a short presentation, please contact me at the following address

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The Teachers as Researchers Working Group of PME

Judy Mousley, Chris Breen, Vicki Zack

The general aims of the *Teachers as Researchers in Mathematics Education* Working Group of PME are to engage participants in discussions about the work of teacher-researchers, to review issues surrounding this work and its contexts, and to facilitate and promote collaborative work in this area.

The group explores the dialectical relationship between teaching and classroom research, in the beliefs that mathematics teachers can and should carry out research in their classrooms and that mathematics educators should research their own teaching and its effects in broader fields. Discussion and other activities generally relate to teaching as a reflective practice and continuous learning process, the nature of the theory/practice interface, the critique and dissemination of research findings from various contexts, the types of research problems being generated in classrooms, and methods of finding solutions within the context in which questions arise.

During the 1994-5 year, members of this working group have been using the electronic mathematics discussion list <mathsed-l@deakin.edu.au> and received two newsletters. A bibliography of readings related to the topic *Teachers as Researchers* is being built up by the Working Group. A first draft will be made available at Recife—for review, as well as deciding on areas needing further input as well as methods of sensible organisation.

The 1995 Working Group sessions will workshop issues arising from case studies drawn from participants' work in this area over the past year. The cases will include:

- examples of participatory research by groups of teachers in schools, with external support;
- a teacher who researched 'blockages' in her pupils' understanding of mathematics and hence developed an understanding of the differing perspective that being a researcher could give;
- an action research project regarding in-service training of mathematics teachers to support the introduction of a new textbook;
- university staff whose work with teachers led them to think about the act of teaching in different ways, impacting on their own (tertiary) practices;
- teachers and researchers working together and using data from classrooms to explore particular mathematical concepts and related psychological models; and
- classroom research using 'noticing' as a methodology.

New participants in the Working Group are welcome to attend these sessions, and encouraged to present a case—with a maximum of five minutes for presentation followed by a short activity-based workshop.

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Advanced Mathematical Thinking

The AMT working group is concerned with all kinds of mathematical thinking, developing and extending theories of the psychology of Mathematics Education to cover the full range of ages. This interest includes gathering information on current research, discussions of both the mathematical and psychological aspects of advanced mathematical thinking, and research into thinking in specific subject areas within mathematics.

This will be the tenth meeting of the working group. Last year our discussions of computers, proof, and the psychology of advanced mathematical thinking concluded with a proposal that we should consider two topics at this years meeting: the formulation of mathematical thinking, and the relationship between social contexts and mathematical thinking.

Essential to the development of mathematical thinking is the transition from informal to formal reasoning. This is the basis of both the concept of proof, developed out of informal deductive reasoning, and transformation of the structure of mathematics itself into objects of mathematical investigation. Informal reasoning, in the form of intuitions and insights, also plays an important role in the creative thinking of professional mathematicians.

It is also increasingly apparent that social contexts are an important aspect of advanced mathematical thinking, and are vital in its development. The mathematical communities of classrooms and the professional mathematical community are two social contexts in which mathematical thinking evolves.

Our first session at PME XIX will include short presentations on the transition from informal to formal, with plenty of time for discussion. Our second session will focus on social contexts, in a similar format. Our third session will focus on issues arising in the first two sessions, especially issues which integrate our consideration of the two topics. We will close with a discussion of future plans for the working group.

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SOCIAL ASPECTS OF MATHEMATICS EDUCATION

PME XIX RECIFE, BRAZIL, 1995

Leo Rogers, Roehampton Institute, LONDON SW 15 5 PH

Session 1: Consideration of aims and purposes of National Curricula.

Some comparisons of mathematics curricula may be made in terms of statements of content, specifying in what year a particular topic appears, and, in some countries, how much time may be given to its teaching. A comparison on the basis of the "intended"; the "implemented" and the "achieved" curriculum may be more useful.

It is intended critically to analyse some current national curricula with a view to identifying the aims, objectives (both mathematical and political) and their social and economic contexts and constraints.

Session 2: Alternative curricula for democratic citizenship.

Many teachers of mathematics would like to see a curriculum which provides the basis for democracy and "responsible citizenship". However, designing such a curriculum is problematic. The "Criticalmathematics Educators Group" group has been active in pointing out the different perspectives of mathematicians, teachers and citizens. This session will look at the options for possible curricula for empowering citizens.

Session 3: Classroom interaction.

Whatever we might do to improve the intended curriculum, the key to success will be its implementation. Here we look at the interrelationships between teachers and pupils; the different modes of discourse, their images of mathematics, their expectations of the system, and their views of the purposes of teaching and learning mathematics.

Session 4: The wider curriculum and future contexts.

What is the position of mathematics in the whole school curriculum ? In many countries it seems to be isolated. Has this happened by its nature, or by bureaucratic expediency ? Is it more "efficient" to teach mathematics separated from other subjects, or should we look forward to a time when mathematics becomes combined with other areas of study ? These questions assume that we teach mathematics in school, and not elsewhere. We look at the increasing role of technology, and wonder if all we need for our daily lives is "push button" mathematics, and that "real" mathematics will only be for a privileged few.

PME WORKING GROUP
CLASSROOM RESEARCH

Organizers: Anne Teppo, teppo@mathfs.math.montana.edu
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The purpose of this group is to examine issues and techniques related to research involving the learner in the classroom.

The focus of discussion at PME XIX will be on research studies conducted in actual classrooms that are investigating children's development of deep conceptual understandings. Each session will begin with a short presentation of one aspect of a study which will provide the springboard for discussion.

Issues to be addressed include; how to create an environment in which deep conceptual learning can occur, the types of research methods that are appropriate for investigating this kind of learning, and how findings based on rich data can be reported.

Short presentations by Leen Streefland (The Netherlands), Carolyn Maher (USA), Tracey Wright (USA), and Miriam Amit (Israel) will lead off each working group session. These presentations will include examples of students' work and video clips of students engaged in the development of deep mathematical concepts within actual classroom environments.

An important aspect of these sessions will be the opportunity for participants to share with each other the ups and downs of research that don't get reported in papers or formal presentations.

DISCUSSION GROUPS

A Science of Need: Exploiting the Powers Students-Generated Constructs and Conceptual Technologies

organizers: Miryam Amit, John Clement, Guershon Harel, Jim Kaput, Colette Laborde, Dick Lesh, Ricardo Nimerovsky

In research on instruction, a disproportionate amount of effort has aimed at finding ways to explain, demonstrate, or illustrate key constructs. However, relatively little effort has been invested in finding ways to create situations in which students confront the need for these constructs.

This discussion group will describe research that has adopted the philosophy that ...
IF YOU BUILD THE NEED FOR A CONSTRUCT, STUDENTS WILL CREATE IT.
The examples that will be discussed will illustrate that, if an instructional setting helps to create the experienced need for a given construct, then even children who have been labeled "below average" often emerge as being remarkably able to develop the construct that is needed. Furthermore, the preceding children often create constructs that are much more sophisticated than those that their teachers had been trying to teach; and, this is especially true if the instructional setting also provides useful tools to facilitate the construction of the targeted conceptual system.

Three closely related types of conceptual technologies will be discussed.

- CONSTRUCTS: These include models, structural metaphors, and other conceptual systems that humans create to structure their worlds ... and to make sense of their experiences by generating useful descriptions, explanations, manipulations, and predictions.

- CONCEPTUAL AMPLIFIERS: These include interacting languages, notation systems, and other concrete/graphic/experience-based representation systems or organizational schemes in which the meanings of the preceding CONSTRUCTS are distributed. They also include other tools, such as those that are based on electronic technologies in which certain human capabilities are instantiated in order to facilitate interpretation, computation, or communication.

- CONCEPTUAL ARTIFACTS: include culturally shared organizational systems (e.g., communication systems, business and economic systems, social/organizational systems, monitoring and accounting systems) which have become some of the most important "objects" influencing people's lives in an *Age of Information*.

In science education research, it is obvious that children create their own constructs for thinking about systems that involve electricity, light, gravity, and magnetic forces, and other scientifically interesting phenomena. For example, in the past decade, analyses of children's *naive conceptualizations* have been among the most productive areas of science education research. However, the emphasis was on ways instruction that could get children to abandon their self-generated constructs; whereas, currently, science education research has shifted toward emphasizing more positive aspects of children's initial interpretations. That is, for the most important big ideas in the school science curriculum, researchers are exploring ways that children's self-generated constructs provide foundations on which more sophisticated conceptual systems must be built.

In contrast to science education, in mathematics education research, a common assumption continues to be that students are generally not capable of creating any constructs (or ideas) of substance. Pundits observe that ... *"When it took professional mathematicians so many centuries to invent the "big ideas" that should underlie modern textbooks, how can students be expected to construct these ideas during a small number of open-ended, unstructured problem-solving activities?"*

This discussion group will involve contributions from the following presenters, each of whom will focus on characteristics of construct-eliciting activities in fields such as: (i) physics (John Clement), (ii) algebra (Jim Kaput), (iii) calculus (Ricardo Nimerovsky), (iv) statistics (Dick Lesh), geometry (Miryam Amit). Special attention also will be given to aspects of construct development that emphasize communication (Colette Laborde) and proof (Guershon Harel). ... The following questions and issues will be given special attention.

- What conditions create the need for some of the most important "big ideas" that should provide the foundations for the school mathematics curriculum?
- How can construct-eliciting activities be used to focus on deep treatments of small number of "big ideas" rather than simply trying to superficially "cover" a large number of idea fragments or isolated skills.
- What kind of construct-development cycles do students tend to go through during the development of useful interpretations and representations of construct-eliciting situations? What mechanisms contribute to these creations or refinements?

POST-STRUCTURALISM DISCUSSION GROUP

Organiser: Paul Ernest, University of Exeter, UK

In the past few years, a growing number of researchers in the psychology of mathematics education have adopted or utilised post-structuralist perspectives. Perhaps the best known is Valerie Walkerdine who gave a plenary talk at PME 14, and is well known for her post-structuralist analysis of young children's language and reason (Walkerdine, 1988). Other recent papers have reflected or drawn upon a post-structuralist perspective, such as those of Jeff Evans (PME 17), Clive Kanes (PME 15) David Pimm (PME 14) and Paul Ernest (PME 15, 17). There was also a lively and well-attended discussion group on post-structuralism at PME-18 in Lisbon, organised by Paul Ernest and Tony Brown.

Post-structuralism offers a number of important insights for the psychology of mathematics education. It stresses the import of social context, that of power and positioning in inter-personal relations, the central role of discourse, language and text, and the problematic and multiple nature of the learner or cognising subject. Currently these issues are topics of central empirical and theoretical interest in the psychology of mathematics education. Thus a continuing discussion of post-structuralist perspective is important and apposite.

This year the group will focus its discussion one theme: **The construction of the subjectivity or subjectivities of the mathematics student as a learner of mathematics**, primarily in the mathematics classroom. How does the learner enter into and participate in the culture of the mathematics classroom so as to become a learner of mathematics? How does the learner interact with texts, and in particular with written-textually presented mathematics tasks? What is the contribution of these to the formation of mathematical subjectivity? How does this lead, in the long run, to the formation of (1) mathematicians, (2) students successful at school mathematics, and (3) students unsuccessful at school mathematics? Are different subjectivities formed? Are the differences in the powers and intuitions developed by the learners related to their positioning in the discourse of school mathematics? How do gender, race and class relate to the positionings and subjectivities so formed?

One theoretical perspective that looks fruitful is that of Rotman (1993) who suggests that the 'doer' of mathematics adopts three different subjectivities. These, in order of decreasing scope, are that of (1) the person, (2) the mathematician, and (3) the agent who carries out the actions imagined by the mathematician. This model has great potential application beyond the fully formed mathematician Rotman considers. The development of these subjectivities is a interesting research issue for the psychology of mathematics education.

The group will discuss what insights a post-structuralist perspective can offer, and how the questions raised might be researched empirically. The group will proceed by short lead contributions followed by open chaired discussion.

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Discussion Group: **"Vygotskian Research and Mathematics Teaching and Learning"**

Convenor: Steve Lerman

The aim of the discussion group is to examine the contribution of Vygotsky and some of his compatriots and the implications of their theory of learning in a socio-cultural context for mathematics education. The assumptions underlying Vygotsky's position differ in several ways from the tacit philosophical and psychological position of the mathematics education community. In particular Vygotsky challenges the centrality of the individual in meaning-making and insists on a social ontogeny of consciousness. By identifying meaning as the unit of analysis for psychology, Vygotsky offered an alternative programme to mentalism, one that focuses on the socio-cultural settings in which the child grows up, on the tools, both physical and psychological, which mediate experience, and on internalisation as the process by which the internal plane of consciousness is formed (Leont'ev). In addition his position transcends traditional Cartesian dualities such as self/other, mind/body, feeling/thinking and subject/object. His historical-cultural method of research differs significantly from predominant methodologies which typically focus on a part of the learning situation.

At PME 18 in Lisbon, the first session developed from some key aspects of Vygotskian approaches chosen and offered by the convenors, and in the second session some research issues/implications were proposed and discussed. At PME 19 in Recife we will aim to continue to interrogate that body of work looking for its relevance for our research, for its complementarity or contradiction to other approaches, and to its deficiences and its strengths. We will also aim to report on some of the current research in the psychology of mathematics education which draws on the Vygotskian perspective and to attempt to identify its particular contributions to research perspectives in the teaching and learning of mathematics.

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Discussion Group Proposal: RESEARCH METHODS IN MATHEMATICS EDUCATION

In the past decade, research in mathematics education has ushered in a series of paradigm shifts; and, related to these, the development of innovative research methodologies, such as: teaching experiments in technology-intensive learning environments, computer modeling of complex problem solving behaviors, sophisticated videotape analysis techniques for real-life problem solving situations, and ethnographic observations in which attention is focused on abilities that go far beyond "shopkeeper arithmetic" from the industrial age.

The development of widely recognized quality standards has not kept pace with the development of new methodologies; and, the result is that naive or inappropriate standards often cause excellent studies to be rejected for funding or publication; or, conversely, potentially significant studies are sometimes marred by methodological flaws.

In the United States, at the National Science Foundation's Program for Research on Teaching and Learning has noted that, because the most innovative and potentially powerful proposals tend to be precisely the ones that stretch the limits of estabished theory and methodologies, they often are the ones that are most significantly affected in negative ways. Researchers are faced with difficult choices. They can use simpler or more traditional methodologies, and perhaps risk compromising the potential power of their research; or, they can use more complex or less traditional methodologies and risk rejection because of limited opportunities to explain their novel approaches.

For the preceding reasons, the National Science Foundation currently is funding a project whose goal it is to:

- Identify a selected number of emerging, nonroutine research methodologies.
- Identify and clarify the theoretical basis for these methodologies.
- Develop guidelines for their appropriate implementation.
- Clarify standards for data analysis and interpretation.
- Identify the practical limitations of each methodology.
- Identify the theoretical limitations of each methodology.
- Situate the methodology within the larger universe of research techniques.

Therefore, we propose a discussion group, that will describe progress related to the preceding issues and broaden the dialog about research methods in mathematics education.

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Embodied Cognition and the Psychology of Mathematics Education

Rafael E. Núñez Laurie D. Edwards
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One of the central goals of the psychology of mathematics education is to understand the thinking involved in doing and learning mathematics. The field of cognitive science constitutes a resource for addressing this goal. Unfortunately, the term cognitive science is generally understood to refer to a particular theoretical approach focused on individual reasoning, often explained in computational terms. There is a widespread belief that in explaining human cognition it is necessary to refer to mental representations, symbol manipulation, and information-processing. These concepts are rooted in an objectivist tradition, in the sense that it is assumed that the objects and world being represented and manipulated pre-exist the knower. The result of this interpretation has been that many mathematics educators, especially those concerned with social and cultural factors, have overlooked the potential contribution of cognitive science as the scientific study of knowledge.

In this discussion group, we intend to challenge this narrow view of cognitive science and to invite the participants to consider new paradigms in cognitive science. In specific, we will focus on perspectives that view cognition as a biological, embodied phenomenon which is realized via a process of co-determination between the organism and the medium in which it exists. We will go on to explore the relationship between these paradigms and the psychology of mathematics education. To do so, we propose to:

- analyze non-objectivist paradigms in different disciplines in cognitive science, including the work of Lakoff and Johnson in linguistics (Johnson, 1987; Lakoff, 1987), Maturana and Varela in theoretical biology (Maturana & Varela, 1987), and Rosch in cognitive psychology (Varela, Thompson & Rosch, 1991);
- explore how these new paradigms can provide powerful tools for both research and practice in the psychology of mathematics education; and,
- invite participants to bring their own research and practice issues into the discussion and to examine them from the perspective of a non-objectivist cognitive science.

We believe that research and practice in mathematics education can only be enriched by drawing from the new perspectives provided by non-objectivist approaches to cognitive science, and the overall purpose of this discussion group is to begin to build a bridge from mathematics education to these new paradigms.

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USING OPEN-ENDED PROBLEMS IN MATHEMATICS

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The method of using open-ended problems in classroom for promoting mathematical discussion, the so called "open-approach" method, was developed in Japan in the 1970's (Shimada 1977). For example in the paper of Nohda (1991), one may find a nice description of the paradigm for the open-ended approach. This discussion group began two years ago in the PME–Japan, and had a continuation in the PME–Portugal (Pehkonen 1994). In these first sessions, the topic of discussions was the concept "open-ended problem" and its classroom usage, with examples from different countries.

In Japan (1993), we concluded that open-ended problems pertain to a larger class of open problems (i.e. problems with openness in the initial or goal situation). Furthermore, open problems contain e.g. problem posing, project work, and most real life problems. The presentations (Nohda, Silver, Stacey) of the discussion group are published (Pehkonen 1995). In Portugal (1994), we continued the basic discussion, and concentrated on the variety of open problems used in different countries.

This year, the discussion group will focus on research results obtained around open-ended problems. There will be 3–4 brief presentations (about 10–15 min) from different countries, containing the presenter's view point of the state of art in his/her country and its surroundings. The presentations will give us some starting points for discussion. The main question will be "What research-based knowledge do we have from open-ended problems?" From this, one could conduct some further questions, as "What recent research has been done on open-ended problems?" and "Are there some underpresented fields on which we should focus our research?".

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SHORT ORAL COMMUNICATIONS

TEACHERS' PRACTICES AND BELIEFS IN A COMMUNITY WHERE HOME MATHEMATICS DIVERGES FROM SCHOOL MATHEMATICS

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This paper reports case studies of two teachers who taught in primary schools in a rural sugar-cane farming community, in the state of Pernambuco, Northeast of Brazil. These case studies where conducted as part of a wider investigation into how children growing up in that community experienced the relationship between their home and school mathematics (Abreu, 1993). When engaged in the practices of sugar-cane farming people make use of an indigenous mathematics, which differs markedly from the "western" like mathematics taught in the local schools.

The neglect of home mathematics by the schools, is a well documented phenomena in Brazil (c.f. D'Ambrosio, 1985, Carraher et al., 1982). The ultimate responsibility for this neglect lies with the teacher who is the mediator of the school culture and fails to establish bridges. However, very little is known about teachers who work in communities where home and school mathematics involve distinct forms of representation. To what extent do they know about the existence of home mathematical practices? If they know, what are their beliefs about those practices? Is there any kind of relationship between their representations of home mathematics and their teaching practices? These are the questions that will be explored in the two case studies presented.

The data for the case studies was obtained in classroom observations, video-tapes of classroom lessons, and semi-structured clinical interviews with the teachers. All data collection was done, in April and May 1991, by the same researcher a native Portuguese speaker, who also had previous experience of research in that community.

Both case studies illustrate that to come to terms with the situation teachers develop representations of mathematics which: (1) enables them to understand and explain the situation and also justify their teaching practices - cognitive function; (2) enables to locate themselves and the children in the social structure of the farming community - social identity function. Nevertheless, the case studies also illustrate diversity among the teachers. Case study 1 shows how teacher's beliefs in the low status of the home practices prevents her from seeing any advantage in bringing home mathematics into school. On the other hand, case study 2 shows a teacher who attempts to valorise home mathematics, but also fails to bridge the two mathematics. In this case due to perceived difficulties in relation to her lack of knowledge of the home mathematics, and also due to the way school mathematics culture is institutionally structured. To conclude we will discuss how these data lead to a theoretical elaboration of the relationship between home and school mathematics in terms of construction of social identities.

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Generalization in Learning Mathematics

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The present study aims at analizing the results of two investigations involving the mathematical competence of adults with limited schooling, as well as offering some thoughts on the limits of these competences. In the studies, the subjects solved problems similar from the point of view of mathematical structure, the contents of which were either familiar or unfamiliar. Eight unemployed adults participated in the first study; all were associated with an institution for adult education in Bourgogne, France. In this study the data collection entailed two sorts of social interaction: researcher-subject and subject-subject. The proposed problems involved percents and proporcionality. In the second study, twenty sugar cane workers from Brazil's Northeast region took part. The proposed problems involved calculation of areas and calculation of averages.

The results underline the limits of generalization in learning mathematics and illustrate some conditions where the transfer of competences has a greater likelihood of occurring. The first study isolated conditions that seem to facilitate transfer: the nature of the social interation, the role of the interlocutor, and the nature of the intervention. In general, pair problem solving enhances performance, except when both problem solvers lack basic knowledge about the mathematical concepts. In these cases, the individuals do not approach the problems mathematically, restraining themselves to talking about the contextual and social background described in the problem. The type of intervention based on oposing viewpoints of the mathematical concept seems to promote socio-cognitive conflict and to enhance performance. The second research demonstrates that transfer can only occur when the problem proposed is recognized as such by the subjects. If this condition does not hold or the problem is considered unsolvable, transfer tends to be weaker or even non existent. These studies show that the elements isolated by the subjects as pertinent for solving problems are not always those contemplated by the researcher. The data offer elements for identifying the competences of subjects with limited schooling, as well as complementary data for analysing the possibility of transfer of these competences.

SOME DIFFICULTIES IN THE DEVELOPMENT OF THE GEOMETRY CURRICULUM ACCORDING TO VAN HIELE

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In the same way as there are elements in favour of the curricular development there are barriers against it as well. As Howson, Keitel and Kilpatrick (1981) state, there are some difficulties concerning the disagreements among different difficulties concerning the disagreements among different ideologies and interests due to political, religious, educational and social causes because all these modifications imply a change in the balance of power; these are causes which affect the educational macrosystem. Others are related to practice itself and they come from the ignorance of the new contents, the lack of appropriate curricular materials as well as the necessity of leaving some contents and some teaching strategies which proved useful and which gave security to the teachers for others that, in any case, imply a risk. These difficulties are related to the educational microsystem.

Thus we can see that the difficulties in the processes of curricular changes affect many elements, but we outstand only three: subject (Geometry), students and teachers.

Researches have been directed more towards the curricular organization of Geometry according to the van Hiele model and towards the change of the students from some levels to others through an instructional design (Clements and Battista, 1992, Jaime, 1993) than towards the problems that come from implementing a curriculum according to the perspective of teachers.

The study we carry out at present makes a part of a wider one (Afonso, Camacho, Socas, 1994) and its aim is to determine the role teachers play in the decision-taking in order to implement a new Geometry curriculum as van Hiele says.

In this paper we will analyse the results of a survey carried out to a group of secondary teachers who have been teaching students ranging from 11 to 14 years old for more than ten years, with the main objective of determine their predisposition to use the teaching proposals suggested in the van Hiele theory. For that reason an experience divided into three stages was designed: In the first stage an open-ended questionnaire was presented through which we determine the amount of coincidence among the ways of developing the Geometry curriculum in the levels in which they appear from their training and experience and what comes from developing the curriculum guided by such a theory of geometrical thought. In the second stage we analyse the training of the teachers according to the van Hiele theory, combining the immersion method with the exhibition of the above-mentioned theory. In the third and last stage we study by making use of a survey similar to the initial one, the rate of agreement of ideas (supposedly modified) that the surveyed teachers had.

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ASSESSMENT AS A LEVER FOR IMPROVED INSTRUCTION - A CASE OF STATISTICS

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The recent changes in mathematics education focus mainly in assessment ways, the leading perception being that "what you assess is what you teach." The traditional method of achievement assessment using multiple choice examinations, in which the student had to cope with a great number of items within a given time, is almost obsolete. These exams reflected the belief that mathematics is a "sterile" subject, unconnected with real life, and the perception that "first you teach and only afterwards you test on what you taught," meaning that the test is the end of the learning process, and not its beginning.

Recent initiatives in mathematics assessment call for a better expression of the application of mathematics in real life and request to do away with the thought that teaching and assessment are mutually exclusive. The linkage approach between instruction and assessment demands a dramatic change in the implementation of task characteristics and in the way students learning is recorded and monitored. In order to meet these needs, a program known as the Packets for Middle School Mathematics was developed in the ETS. This program incorporates Performance Assessment activities, it strongly reflects the NCTM standards and facilitates the delivery of interdisciplinary instruction, cooperative learning techniques, and the application of mathematical thinking in real-life situations. Each Packets activity was designed to elicit construction of mathematical models, based on research by Lesh and others.

In a research carried out in two grade-7 classes, the students were asked to carry out a performance assessment task that required information processing of attributes of consumer goods, and the creation of a model of rating system to be introduced in a consumers guide for the young. The students worked in groups and were asked to document their solution to the task, describe the model they constructed, present it in front of the class and defend it against criticism of their peers. In spite of the fact that this was the first time they had to cope with this kind of task, and despite the teachers' apprehension ("they never learned this kind of stuff"), the students succeeded in creating models based on mathematical and geometrical concepts. An interesting phenomenon was the "invention" of statistical terms equivalent to weighted average, mode and median. These were never learned before, were intuitively developed by the students, and constituted a starting point for the methodical learning of descriptive statistics.

In the session we will present the activity, discuss the linking process of assessment and instruction, and deal with the issue of the use of assessment as lever for further learning.

PLAYING IN THE STOCK MARKET WITH A SPREADSHEET[*]

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Learning mathematics is not merely to assimilate a set of procedures and techniques which would aloud the students to solve a certain type of exercises or to prepare them to learn more and more sophisticated techniques. The integration of applied mathematical problem solving activities in mathematics teaching is framed by a perspective which considers that learning mathematics is also learning to apply it, relate it with other sciences and solve realistic problems.

The use of a computational tool as a spreadsheet is considered to be an important conceptual element in mathematical modelling given the learning processes that are promoted (Matos, Carreira, Santos & Amorim, 1994). Multiple representations of a given situation are made accessible and the user can manipulate and relate them. The computational tool is seen as a conceptual amplifier (Lesh, 1987; 1990) or a reorganizer (Pea, 1987) of the mind.

In this paper I will describe and analyze an experience with 11th grade students exploring theoretical (mathematical) models which describe the evolution of the quotation of the share of stock market. A spreadsheet was used as a working tool. Data was collected through observation and video recording of one group of students at work. Data analysis was a kind of *reconstruction* from the most significant episodes, regarding the role of the spreadsheet.

One of the main results of this study points to the idea that the spreadsheet plays the role of as conceptual amplifier or reorganizer of the mind, depending on several factors such as the students' personal characteristics, the nature of the problem, the kind of interactions between the students and the interactions with the computational tool.

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ON THE DIDACTICAL TRANSPOSITION OF CARDINALS AND ORDINALS[1]

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According to Zermelo-Fraenkel set theory, at least in the version of Halmos (ZF-H) cardinal and ordinal numbers can only be distinguished in the infinite case: *"(...) if X is finite (...) the cardinal number of X is the same as the ordinal number of X"* [Halmos, 1960. p. 100]. Also, according to Piaget, *"(...) finite numbers are simultaneously cardinals and ordinals; this results from the very nature of number as a system of classes and of assymetrcial relations, merged together in one operational whole"* [Piaget, 1975, p. 219]. Hence, mathematically (ZF-H) as well as psycologically (Piaget), finite cardinal and ordinal numbers coincide. Nevertheless, cardinals and ordinals are culturally recognized as distinct mathematical notions and are taught as such in elementary school. According to Chevallard's Teory of Didactical Transposition (DT), in order that a specific knowledge can become an object of teaching, it must have *educational* and *epistemological legitimacy* [Chevallard, 1989, p. 63]. In so far as social agents in the teaching sphere recognize ZF-H and Piaget as the basic references for the teching of C&O, a legitimacy problem arises and takes the form of a contradiction of DT: elementary teaching deals with two distinct notions that, according to scientific knowledge, should be considered the same. We have investigated how people live along with this contradiction in the different levels of teaching. We have visited elementary school classrooms, examined official sylabuses and textbooks and interviewed mathematics teachers of all teaching levels. According to Arsac [1992] and Assude [1992], we have considered three possible modes of DT: natural DT, counter DT and a blockage of DT. Our results so far indicate that the blockage mode is the dominant one; teaching ordinals tends to be left to the Language teacher.

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[1] This communication condenses results af a research leading to the M.Sc. degree of the second author who has the first one as advisor and the third one as co-advisor.

SUPPORTING GRADUATE TEACHERS TO CARRY ON INNOVATIONS: A DESCRIPTION OF HARRY'S ATTITUDE

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This paper describes and analyzes the attitude of a Professor who decided to adopt a new pedagogical approach for his Linear Algebra class. How he decided to change his way of teaching, what kind of support he got, and what expectations he had.

The results suggest the possibility of changing in mathematics teaching and learning by supporting teachers to carry on innovations.

INTRODUCTION

The implementation of a new methodology in Linear Algebra grad class was the initial aim of the research, but my attention swerved to study, in depth, teacher's attitude towards implementing an innovation.

The importance of this study relies on the fact that we usually plan on supporting elementary and secondary teachers to carry on innovations but that is not common to do the same with Professors. Also, a result of this research reinforced that as happens in the agriculture or economic scenario a person should have a strong support in order to move from adoption to implementation and from this phase to routinization.

FIELD WORK

This study took place in a Math Education master's course, in Rio/Brasil. Harry, a pure math oriented teacher, uses to teach Linear Algebra and other math courses for undergrad and grad students, in the same way probably most of math teachers do. After a seminar about Writing in Math Classroom (given by Arthur Powell, Rutgers University) he told us about his deep interest in trying this new approach and how he was willing to start a different course but he would like to get a " real support" for this task. Figuring out that it would be the perfect opportunity to meaningful changes take place we planned the first lesson together and decided to meet one hour every week to read the students production, to comment on them, to reflect about what was going on, and to plan the next lesson.

The classes were divided in two parts: one was the writing part and the other the content part. In the first part, each group of three students (A) chose a "math text", commented on it, gave it to another group (B) to comment and reply B's comment. Harry's task was to write them back in order to improve their mathematics contents. How to do it without giving the right answer? How to keep students motivated to go further in the math topic?

During one term (a semester) we were able to meet once a week for 1:30 hour, during these meetings I took notes, I interviewed Harry regularly and a few students not so often, and I made cópias from the students work. By the end of the term Harry answered a questionnaire and talked about his answers in an interview.

Conclusion

As expected, Harry was divided in adopting a different approach and lecturing. Although he wanted to change he worried about the syllabus, so two of the classes were only lectures. He is willing to use the writing approach again, because he said that some difficulties and mistakes he was able to detect from the students'writing he could not detect if it were in a regular exercise.

It was an opportunity to rethink teacher's role. He felt that contrasting to the lecture approach, using the writing approach he was more like *a coach, sort of a mixture of advisor and a critic*.

PROBLEM SOLVING PROCESSES INVOLVING THE CONCEPT OF DIRECT PROPORTIONALITY AND DIDACTIC PERSPECTIVES OF MATHEMATICS PRE-SERVICE TEACHERS

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Proportionality is a concept of several day-to-day and academic applicability. Nevertheless it can be verified that, faced with problems involving such a subject, students either demonstrate a great deal of difficulty in associating and/or determining the necessary algorithm and/or heuristics to be used, or opt (mainly in extra curricular situations) for a far less formal approach vis a vis the one elected by teachers in tackling the same subject. From this point of view some researchers are of the opinion that integration, in the teaching-learning process, of a large variety of problem solving processes of this type, including the less orthodox ones, contribute towards the construction/solidification of a better conceptual knowledge. For this to be plausible teachers themselves have nevertheless to have the aptitude to use such a variety of strategies.

Given this context we conducted an "exploratory study" with the aim of analysing the performance of 10 Portuguese pre-service mathematics teachers (grades 7/12) as to the individual ways they solved problems involving the concept of direct proportionality when asked not to use (N) or to use (Y) a variety of processes and analyse the strategies these pre-service teachers say they would use in dealing with such problems in situations (N) and (Y).

Based on data collected with the instruments we used (5 problems - where one was accompanied with several resolution proposals - and 2 questionnaires developed for the purpose) we conducted a study of qualitative nature with descriptive, analytic and interpretive intent which allowed us to draw as main conclusions that: (1) participants evidenced great difficulty in the acquisition/ application of the concept under study; (2) they showed a preference for the resolution and for the teaching of problems in which the unit value was stated, by using the strategy which resorts to the cross-product algorithm; (3) they showed a preference for the resolution and the teaching of problems in which the unit value was not stated, by first electing to seek its determination, using less formal processes than the ones used for tackling the subject "proportionality"; (4) although evidencing sensitivity to the need of approaching problems by more than one process, in situation (N) participants merely restated the processes they had used, essentially algebraic, or by means of schemes/ tables. In situation (Y) there were no significant differences.

STUDENTS' ACTIVITY IN MATHEMATICAL MODELLING: WHERE DO THEIR MODELS COME FROM?*

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When speaking of mathematical modelling of real situations we often come across the idea that it is a process by which certain aspects of reality are isolated and translated to mathematical structures. This way of conceiving mathematical modelling also tends to see mathematics as the main source of insights and conceptual tools for the *adequate* representation of empirical phenomena.

What we wish to emphasize in this presentation is the insufficiency of this kind of premise to describe and understand students' activity during their process of mathematical modelling. In our approach we are assuming that a mathematical model is not a model of reality but a model of a conceptual system based on a certain interpretation of reality (Skovsmose, 1990). We will also look at students' activity from a sociocultural point of view, specially in what concerns the mediational means that shape students' actions and models (Wertsch, 1991). Moreover we will stress the role of students' previous ideas and the situated nature of their cognitive processes that make their modelling processes quite different from those of scientific experts (Driver, Guesne & Tiberghien, 1990).

We shall focus on our recent research with a class of 10th grade students that were involved in mathematical modelling situations and had the possibility of using a spreadsheet to work on the problems presented. While reporting the activity of a group of students, dealing with the problem of modelling the cost of a taxi run, we will outline and discuss some results:

a) Students' modelling activity reveals a particular situated nature where previous ideas about the real situation play an essential role; the need to adequate these early interpretations to mathematical structures and patterns seems to be far more located than structural in students' processes.

b) Students' conceptualizations may be contradictory but still very stable; although students describe the discontinuities of the taximeter readings, in successive periods of time, they adopt a continuous linear model to describe it mathematically.

c) The tools that students are provided with can be very strong factors in mediating their activity; when trying to build a general model for the price of a taxi run students first looked for an equation and then struggled with the *need* to use the spreadsheet which proved to be both an obstacle and a powerful representational tool.

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PUPILS' IDEAS ABOUT MATHEMATICS

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When pupils arrive at school they already have their own ideas about Mathematics as well as some knowledge related to it. On one hand, Mathematics knowledge is exterior and pre-existent to themselves. On the other hand, they had previous Mathematical experiences in their daily life. Anyhow, school Mathematics quite ignores pupils' reasonings in daily life Mathematics (Carraher, Carraher & Schliemann, 1989; Saxe, 1989).

The didactic contracts established at school play a main role in the ideas that pupils build up about Mathematics (Schubauer-Leoni, 1986). The social construction of the meanings in a Mathematics class is deeply related to the interaction among the didactic triad: teacher, pupil and knowledge (Perret-Clermont & Schubauer-Leoni, 1988). Pupils are not mere reproductors of schemes and practices. Therefore, the way they conceive Mathematics deeply influences their relation to this school subject. Moreover, the understanding of pupils' representations about Mathematics is specially important due to the fact that underachievement is quite high in this school subject.

As part of a broader study, pupils from the 7th level (N=331) at Lisbon were asked about their ideas about Mathematics -- whether they liked this school subject or not, and why. Most of the pupils related Mathematics to computation. This was true both for those who liked it and those who didn't. Some of them also described it as useful for their future or for daily life. The main difference between those who liked it and the those who didn't is that the first ones said it was interesting and funny whether the others perceived it as complex and difficult. Only very few pupils associated it to problem solving or to a "mental game" which reflected that these students usually had a traditional conception of Mathematics and related tasks in a school class. These findings are consistent with previous international research by Lapointe, Mead & Askew (1992). Anyhow, the most surprising result was that a lot of pupils reported that Mathematics was related to "memorizing things".

YOUNG CHILDREN'S USE OF SHARING AND THEIR UNDERSTANDING OF THE RELATIONSHIP BETWEEN DIVISION TERMS IN A NON-COMPUTATIONAL TASK

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In a previous study (Correa, 1994; Correa and Bryant, 1994), 5- to 6-year-old children were asked to make judgements about the relative size of the quotients in non-computational division tasks The analysis of their informal strategies for solving concrete division tasks revealed that most of the successful children made use of correspondence procedures. We decided to investigate whether children's immediate experience of sharing can help them to work out the inverse divisor-quotient relationship in non-computational tasks. Our hypothesis was that asking children to share the quantities between one element of the pair of divisors would encourage them to think about the correspondence between quantities and consequently about the relations (same, more or less) to be established between divisor and quotient. We gave this experience to an experimental group, but not to a control group, and looked at the effect in partitive and quotitive tasks.

The experience of sharing had only a limited effect on 5-year-olds' judgements, even though they are at an age when children perform well in sharing tasks, and this effect is restricted to partitive tasks and is not always accompanied by the proper justification. In contrast the experience had a much larger effect on the 6-year-olds judgements about the inverse divisor-quotient relationship, both in terms of correctness of responses and in terms of the explanations given. The analysis of the children's justifications also leads to a hypothesis about the cognitive mechanism involved in the solution of the task. It is possible that the main cognitive mechanism used by children for solving the task involved the mental redistribution of the quantities in the form of new equivalent sets. The children in the experimental group used the result of their active sharing as a baseline from which the quantities were mentally redistributed into more sets (leading then to a smaller quotient) or, on the contrary, into fewer sets (resulting into a quotient of greater size).

Two major conclusions can be drawn from these findings: a)proficiency in sharing tasks does not immediately lead to a comprehension of the relationship between the quantities in division; b)correspondence procedures (active sharing) can be used to help some children to overcome some of their difficulties in judging the correct relationship between division terms in non-computational tasks.

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NEW SCIENCE, COMPLEXITY AND THE MATHEMATICS CLASSROOM: EXAMINING RELATIONSHIPS, NOT THINGS

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New Science approaches complexity by rejecting reductionism, fragmentation, and notions of control characteristic of the Newtonian worldview and examining relationships and interactions rather than entities or things. Organizational theory, sensitive to the adaptive nature of organizations, considers the dynamic nature and formation of relationships. Complexity theory applied to learning organizations offers insights into the precarious balance between individual personal autonomy and organizational life by adopting the perspective of New Science.

This short oral report will discuss visions of schooling constructed by 67 preservice teachers with regard to the complex relationships which define the mathematics classroom learning environment. Using Habermasian critical philosophy and metaphor analysis to identify preservice teachers' perspectives, the teacher-students, students-students, and teacher-self relationships were examined. The guiding question for this investigation was:

> How do metaphors describing the teacher-students, students-students, and teacher-self relationships in the mathematics classroom contribute to understanding preservice teachers' perceptions of classroom dynamics from a New Science perspective?

The overwhelming majority of preservice teachers participating in this investigation described a vision of schooling whereby all interactions and communications were through the teacher. Students were perceived as isolated objects and not part of the classroom context by virtue of their relationships with each other or their individual quests for understanding. The perspective of these preservice teachers was not consistent with the holistic vision of learning organizations as these preservice teachers described students as 'things' acted upon by the teacher.

Information and relationships, rather than entities or 'things,' are the basic elements of Information Age dynamics. Schools must adopt a New Science stance or else the ever-increasing complexity of the classroom combined with the increasing fragmentation and bureaucracy of schools will further limit students from interacting meaningfully with mathematics and lead to continued alienation from and fragmentation of mathematics learning. Research consistent with this vision of the future will be explored in discussions associated with this short oral report.

A STUDY OF TYPES OF PROOFS PRODUCED BY STUDENTS OF THE SECONDARY SCHOOL ON THE DOMAIN OF WHOLE NUMBERS

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Université René Descartes - Paris - France

In the secondary school teaching through demonstrations is one of the important objectives of mathematics classes. Geometry problems are often used to teach students how to produce proofs. Thus, Balacheff (1988) has proposed in this field a typology of proofs, based in the work of secondary school pupils. In increasing order of complexity, four types of proofs are distinguished classifying into two higher-level categories: pragmatic proofs and intellectual proofs.

A legitimate question which can be raised is whether this model is transferable to other areas of mathematics. The present study examines the productions of eighth and ninth grade students, to the types of problems studied at the turning point between arithmetic and algebra, on the domain of whole numbers. In comparaison with Balacheff's typology, in our experiments with the students solving several arithmetic-algebra problems, we only encountered one type of pragmatic proof, while our intellectual proofs were more diversified than in the model. In order to account for our experimental data, we propose a three-class typology with one type of proof at the pragmatic level and two at the intellectual level. Thus, we are encountered the following types of proofs: **Pragmatic proofs**, are based on numerical calculation; **Statement proofs**, proof mad up of an organized sequence of elementary statements in natural langage and **Algebraic proofs**, that consist of validating statements using algebraic terminology.

We conclude that for intellectual proofs the typology proposed by Balacheff must be adapted to arithmetic-algebra problems.

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A DIFFERENT APPROACH TO ALGEBRA AND PROOF: BEHAVIOURS OBSERVED IN CLASSROOM

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In this oral presentation we report the main lines of a project addressed to students aged 16 in which a didactic path is developed starting from theoretical mathematical aspects (in number theory) to applications (cryptography). The ambition of this activity would be to make students reach awareness of the way of working of the mathematician (within limits of the school level involved); a particular emphasis is given to activities such as the production of conjectures, counterexamples to facilitate the transition to formal proof (Moore, 1994). In pursuing this task we were helped by the use of the computer.

As a context for our work we chose the first elementary topics of number theory for these main reasons:

- students consider the set of natural numbers a domain familiar because have worked in it since the first school years
- the statements are simpler to read than those, for example, of geometry (usually they are quite short and hypotheses and theses are better singled out)
- it is possible to find interesting applications, such as cryptography, within students' reach
- it is made easy by the context to revisit algebra from a different point of view.

The context of the experience provide elements to study the educational issues emerged according to different streams: in (Furinghetti & Paola, submitted) the impact of computer in this experience is analysed, in (Paola, to appear) the mathematical content introduced is presented. Here we consider the learning problems observed as for algebra and proof. They may be summarized as follows:

- difficulties in producing formulae
- difficulties in giving a sense to formulae and to control them
- difficulties in the individuation of the context of the problem
- difficulties in representing the reasoning through the algebraic manipulation
- difficulties in interpreting the statements proposed.

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DIMENSIONS OF MUSIC IN EARLY CHILDHOOD MATHEMATICS

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This paper explores how elements of music education might provide effective aids to higher achievement in mathematical development in early childhood. Recent research investigating the development of early number concepts and processes has been influenced largely by the constructivist view of learning (Steffe, 1990). In consonance with the constructivist paradigm, early childhood educators portray children as active thinkers, who construct sense and meaning out of personal practical experiences. Making sense is the purpose of education; to encourage children to look for similarities, oppositions and connectedness in the sensory inputs they are receiving (Dienes, 1987).

In the learning of mathematics, the ability to solve problems is considered one of the most important skills for children to develop. In early childhood, problem solving is really creative problem solving in that it requires a wide range of creative, conceptual and logical thinking abilities to combine in reaching a solution. Early childhood music provides contexts where creative, conceptual and logical thinking combine as windows of opportunity to develop and reinforce mathematical concepts. Many people believe that the learning of mathematics and the learning of music are related but there is little evidence to make such convictions persuasive arguments. Efforts to integrate the teaching of mathematics with music are rare indeed (Kleiman, 1991). Research has implied that a group of children with extra music training provide more creative, original and complex ideas and a higher level of abstraction than those with the usual amount of music (Kalmar, 1989) In brief, mathematics is an aesthetic and creative (artistic) endeavour; and music offers viable opportunities for the development of mathematical concepts in young children.

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Students' representations of rates of change

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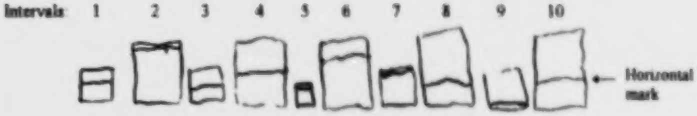
The research reported in this presentation aimed at analyzing 5th, 7th and 9th grade students' understandings of the concept of rate of change and their ability to invent graphs to represent it (diSessa et al., 1991). The study focused on students' developing competence at building adequate paper and pencil representations of rates presented in five situations: plant growth, filling of bottles, displacement of accelerated cars, frequency of a pendulum, and frequency of waves in a beach. These situations were presented in tasks involving: (a) tables representing several consecutive intervals with distinct rates; (b) comparison of rates between two displacing objects; and (c) continuous Cartesian graphs. The combination of situations and type of task was such that all involved contextual clues and numerical values to allow the computation of rates, as well as the production of diagrams. The figure below shows the graphical representation built by a 7th grade student based on a table of the velocity of a car along ten consecutive intervals with distinct rates. The area of each rectangle in the graph shows the duration of the time interval (given in hours), and the height of the horizontal mark shows the distance traveled (given in kilometers).



The diagram above allowed the student to construct a relation between area of the rectangle and height of the mark in such a way that he could easily recognize, for instance, the interval where the object traveled fastest (the smallest rectangle and the highest mark in the fifth interval implies the shortest time and the bigger distance traveled, respectively). Many such idiosyncratic representations were invented by the students.

The overall analysis showed that students make extensive use of representational objects acquired in activities outside school (e.g., reading of newspapers and comic books) in order to represent rates of change in the situations given. Furthermore, the types of representations built by the students are intimately related to the task situation, and their definition of the concept of rate. This communication will analyze the various types of representations constructed by the students, and suggest a developmental trend for the acquisition of this concept.

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Social interaction and mathematical discourse in the classroom

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A set of variables for coding and analyzing mathematics classroom interactions is proposed. These variables are designed to take into account the social interaction between the actors (teacher, student on the blackboard, and group of students), the mathematical discourse itself, and the contributions that each actor makes to the mathematical discourse. The interaction between teacher and students takes place during a lesson *segment*: a block of time with a distinct pattern of activity during which a mathematical *task* is worked out.The set is divided into two groups: one for the coding and analysis of social interaction and another for the coding and analysis of the mathematical discourse that takes place during this interaction.

The variables proposed to analyze the teacher performance during a segment are characterized according to the teacher's contributions to the discourse, the way the teacher handles the students mistakes and contributions, and her/his own authority concerning the validity of these contributions. The variables proposed to analyze the students' performance are characterized according to their reaction to the teacher's contributions, their attitude towards the teachers' authority, their contributions to the discourse, and the way these contributions are accepted as valid within the interaction.

The mathematical discourse that takes place during the development of a task in a segment is characterized by the following dimensions with their corresponding components: representation systems used (verbal, symbolic, graphical, tabular), knowledge types involved (conceptual, procedural), physical mediums involved (verbal, written on the blackboard, written on paper), resources used (textbook, calculator, overhead projector), and actors contributions (teacher, student on the blackboard, group of students). A component is said to have been *present* in the solving of a task within a discourse if that component was used or involved in that discourse. A component is said to have been *necessary* for the discourse if the elimination of its use or involvement renders the discourse meaningless and the task cannot be accomplished. Finally, a component is said to have been *sufficient* for the discourse if the use or involvement that could have been done of the other components of the corresponding dimension were not necessary.

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[*] The research reported in this paper was done in a collaboration project between "una empresa docente" of the University of los Andes and the Department of Didactics of Mathematics of the University of Granada (Gómez, 1994). It was supported by COLCIENCIAS, the *Fundación para el Avance de la Ciencia y la Tecnología* of the Colombian Central Bank, and Texas Instruments.

ARE TEACHERS PREPARED TO INTRODUCE THE CONCEPT OF VOLUME MEASURE?

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This communication contains a partial description of results belonging to a wider research work on the improvement of the abilities of visualization and on the use of the van Hiele Model, aimed at the understanding of the defficiencies inherent to prospective and in-service secondary teachers in learning and teaching geometry.

During the past five years, while examining brazilian textbooks, we noticed that in some of them use has been made of perspective drawings of rectangular solids, as those described by Ben-Chaim, Lappan and Houang (1985), to introduce the concept of volume, even to children below ten years of age.Our concern with that situation, similar to those described by Ben-Chaim et. al.(1985 and 1989) among others, regarding the difficulties faced by students of ages 10 to 13 with the description and the representation of solids composed by piled-up cubes as well as with the interpretation of the drawings used to represent such solids, led us to promote, following the line of our research, a study involving the performance of adults in interpreting the drawings mentioned and in evaluating the related volumes. Particularly, we were interested to find out wether teachers were aware of the conventions implicit in the drawings and of the role of concrete materials in the process of understanding these conventions since brazilian textbooks offer no explanations about the compositon of the drawn objects.

The study was undertaken in three steps. To start with, we developed a set of written tests intended to measure the extent to wich the subjects would correctly interprete drawings and evaluate volumes. In the sequence we developed a teaching unit intended to promote the acquisition and the development of adults' visual abilities. To complement, we performed an analysis of the effects of the application of the teaching unit via individual interviews held with undergraduate students.

The written tests, which were applied to 590 subjects, included, among others, some questions of the National Assessement of Educational Progress Test, 1977-78 (Ben-Chaim et al, 1985). The analysis of the results indicated a rather concerning picture since scores were far below the expected averages. For example, correct answers reached by in-service teachers in the determination of the volume of a simple rectangular prism scored no more than 57% for women and 84.5% for men.

The teaching unit, similar to that proposed by Gutierrez (1992), was based on the use of the van Hiele Model of thinking, and has been applied in teachers' training courses since 1991. Among the effects of the application of the teaching unit and as confirmed by the interviews held, one was that of teachers expressing that only after manipulating with the cubes they became aware of the conventions implicit in drawing and interpreting the representative drawings used; besides this, a remarcable improvement could be noticed in the quality of the drawings made. Moreover, we can report that after finishing the teaching unit, teachers became more confident in their own performance, far more receptive to understand and analyze children's difficulties and strongly motivated to apply construction activities with concrete materials prior to developing the activities proposed in textbooks.

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MODELING EFFECTIVE PRACTICE AND PROMOTING REFLECTIVE PRAXIS: USING INTERACTIVE VIDEODISK IN TEACHER EDUCATION

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This study investigated how use of an interactive videodisk information system (the Strategic Teaching Framework—STF) helped preservice teachers (PSTs) expand their visions of teaching, learning, and assessment in mathematics and their skills in translating that vision into action in the classroom. The subjects in this study were 16 PSTs enrolled in Indiana University's Elementary Certification Graduate Program (ECGP). Use of STF was a new but integral part of a required early field experience where PSTs traditionally spend one day per week for about 12 weeks working with a teacher in an elementary school classroom.

One component of the STF system is a collection of videodisk examples of the mathematics teaching of three exemplary elementary school teachers, along with commentary on the videos from the perspectives of the teacher, a mathematics educator, and the STF developers. A second component of the STF information system is a conceptually organized information base, including resources on topics such as assessment, management, teaching strategies, problem solving, and planning.

For the first four weeks of the term, PSTs spent one afternoon per week at the university working with STF and designing lesson plans, and one morning per week either in teaching workshops (during first two weeks) or in work in their field experience classroom (during the 3rd and 4th weeks). From the 5th to the 13th week, the PSTs spent an entire day each week working with a teacher in the school, as well as carrying out four assignments: (1) conducting a series of interviews about mathematics concepts with three children from diverse levels of ability, and (2) - (4) designing and presenting three mathematics lessons to the class as a whole.

We collected data to document how PSTs used the STF technology in formulating personal visions of effective teaching, how they planned and taught mathematics lessons to actual classes of children, how they critiqued their own teaching and the teaching of others, and how their philosophies and beliefs about teaching and learning developed throughout the course. Data on use of STF came from observations, student journals, and computer logs of the frequency and types of use. Effectiveness data were derived from student belief and attitude surveys, weekly student journal entries, and documentation of the students' performance both in actually teaching and in the analysis of teaching. In several cases, the lessons that the PSTs taught were explicitly modeled on lessons viewed in STF and they reflected thoughtfully on their adaptation of the STF models in their subsequent self-critiques. The PSTs evidenced many changes in their views of mathematics and effective teaching over the course of the semester. While it is admittedly impossible to isolate the influence of STF from other influences (methods instruction, class discussion, school observations, etc.), our research provides compelling evidence about the various ways that STF contributed to the PSTs' growth.

THE STABILITY OF ALTERNATIVE PROBABILITY CONCEPTIONS

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As reported at PME 17 (Laridon and Glencross, 1993), research uses questionnaires adapted from Green (1988) to investigate the understanding of probability amongst grade 9 pupils (14 - 15 years of age normally) in the Witwatersrand and Transkei areas of South Africa. This questionnaire has been applied to about 1 200 pupils.

A more detailed analysis is presented of two series of items from Green's questionnaire. A novel sequential path approach to the statistical analysis of these series of items is used. Alternative conceptions are posited as being the basis on which pupils made choices amongst the alternatives presented with each item in the series. The group of pupils fitting into a particular conception as indicated by a choice is followed through the series. Some startling results emerge in terms of the actual final percentage of pupils who have used a particular conception consistently throughout the series. This cascade analysis is illuminated by an analysis of accompanying free responses. The categories obtained are discussed in terms of possible underlying causes as found in the literature (Borovcnik and Bentz, 1991). The outcomes of this analysis, questions statements often made about the stability and persistence of alternative conceptions. Serious doubts are also raised about the reliability of the usual statistical analyses of instruments consisting of multiple choice items.

Statistics relating to Green's probability concepts level for the sample as a whole and for some sub-samples will be compared with similar statistics obtained from studies in Canada, Hungary, Brazil and Britain.

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"Algebraic" word problems and the production of meaning: an interpretation based on a Theoretical Model of Semantic Fields

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Geraldo G. Duarte Jr.*

In this paper we present both theoretical and empirical support to a view on people's cognitive functioning. In particular, we argue that people's approaches to the solution of "algebraic word problems" is crucially related to the objects they constitute and deal with in that process. The key notions are those of knowledge, Semantic Fields, objects and interlocutors, notions which are constituted as part of a *Theoretical Model of Semantic Fields* (**TMSF**) (Lins, 1992, 1994). From the point of view that theoretical framework, an interpretation of data drawn from two interviews is produced.

Within the **TMSF**, *knowledge* is understood as a pair (*statement-belief, justification*), rather than as in the traditional view, according to which "knowledge" has the status of a proposition, "that which one knows to be the case" (cf. Chisholm, 1989). For instance, according to the traditional view "2+3=5" would be "knowledge." But within the **TMSF** this is a *text*, not *knowledge*. The enunciation of *knowledge* establishes three things: (i) the subject believes in what he is stating, implying that, (ii) he believes to be entitled to have that belief; and, (iii) it constitutes *objects*.

The other key construct in the **TMSF** builds precisely on the constitution of *objects*, and provides the means to account for that process in a more general way. Any *Semantic Field* has a *kernel*, in relation to which particular *objects* are constituted. In the case discussed above, one might speak of a *Semantic Field of fingers*, possibly with a *kernel* consisting in two hands, "concrete" or schematic. Within that *Semantic Field* one can constitute *objects* and operate on them; numbers from 1 to 10 would be such *objects*. Other *Semantic Fields* could develop around a scale-balance, wholes and parts, function machines, areas, money, *number* as a measured collection, or Algebraic Thinking. *Knowledge* is always enunciated to some *interlocutor*, who may be physically present, remotely present or even fictional; the *interlocutor* may be "internal" or "external." It is the subject of *knowledge*'s expectation that his *interlocutor* will be able to: (i) produce *meaning* for the *text* of his *statement-belief*; **or**, (ii) produce *meaning* for the *text* of his *statement-belief* within the same *Semantic Field* from which the *knowledge* was enunciated.

Two Brazilian sixth-graders (12-13 years) were asked to solve verbal problems which potentially involve the manipulation of "algebraic relationships." The scripts and the transcription of the video tape were then analysed in order to determine: (i) the *objects* with which pupils were dealing; and, (ii) the role played by choices of *interlocutors* both in the solving process and in the subsequent attempts to explain what they had been doing.

A FULL VERSION OF THE PAPER WILL BE AVAILABLE TO THE PARTICIPANTS OF THE SESSION.

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COLLABORATIVE RESEARCH ON TEACHER'S TRAINING,

S. M. de Katz, S. Visciglio, M.Castelaro, M. Mas. **UNIV. NACIONAL DEL LITORAL**

This work is a collaborative research (1) intended as a contribution to secondary school teacher's training. The research was carried out by university teachers in charge of teaching algebra to students initiating their business studies at the Universidad Nacional del Litoral, Argentina. The working hypothesis is that student's errors are closely related with strategic and didactic difficulties of teachers. The work design was based on: 1) administering a test to: **Students (A)** initiating their university studies; **Solvers (R),** Secondary school teachers; **Qualified Monitors (IC),** Secondary school teachers having enough experience to answer the test questions as their students would; 2) defining a democratic reflection environment for the teachers who participate which enabled them to correct and analyze their own tests, comparing their results with those obtained by the students; 3) issuing proposals and recommendations. The test includes 36 items (divided in 4 parts) including both issues of syntax and semantics in algebra (2) , as well as the quantitative and qualitative interpretation of graphs to assess the level of "elementary mathematics from the adult's viewpoint" (3). The test was taken by 652 students and 40 voluntary teachers working at secondary school where most (80 %) of the tested students come from (22 **R** and 18 **IC**). The statistical report of the results, Variance Analysis (Friedman) with significance < 1% shows that **IC** behavior is similar and that the **A-R,** but **IC-R** behavior differs throughout the test. In the Correlation Analysis (Sperman and Kendall) there is a significative correlation of scores in 3 groups with a significance smaller than 1% in all cases. Interpretation of the results in **A** and **R** is similar to those reported by Hershkowitz, M. Bruckheimer, and S. Vinner (1987), Chapter 19 in Learning and Teaching Geometry, K-12. The information on the errors the 3 groups made and the strategy of sharing reflection analysis and self-criticism allowed to detect examples of over-and under-estimation of students' learning as well as to identify didactic difficulties and the need of re-examine concepts such as "variable" (4) and analyze "Symbol Sense" (5). The stimulus originated for teacher's training leads to an alternative proposal for the curriculum design.

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ETHNOGRAPHIC RESEARCH METHODOLOGY AND MATHEMATICAL ACTIVITY IN THE CLASSROOM*

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Mathematics education research is becoming increasingly concerned with the study of classroom processes and the complexity of interactions within them. This increased interest in the sociocultural aspects of the classroom work has led to a shift in the nature of research undertaken. It recognizes the importance of social interactions within classrooms and what students and teacher bring to them.

The focus of our project is to investigate *what is* mathematical activity in the classroom from the point of view of the participants. We assume that theory is essential in order to *learn* from the students and teacher about their mathematical activity. The conceptual framework takes relevant notions together from Jean Lave's ideas of context and activity — persons-acting, arena and setting being implicated together in the very constitution of that activity (Lave, 1988) — and Vygotsky's idea of mediation. Mediators enter into the organization of behavior in two ways that underline their conceptual and material nature as they act simultaneously as tool and constraint — in coming to master aspects of the world, children come to master themselves (Vygotsky, 1978).

In order to investigate students' classroom activity, an ethnographic research methodology is adopted. On this paper, it is discussed the adequacy of an ethnographic approach — the art and science of describing a group or culture (Fetterman, 1989) — to the study of students' mathematical activity. Considering the idea from Spradley (1979) that rather than *studying people*, ethnography means *learning from people*, we elaborate on the use of an ethnographic approach to investigate classroom activity which is conceived according to three principles:

(i) activity is situationally specific and this implies that objects of analysis are points of cultural-historical conjuncture and should be analyzed in those terms; (ii) to focus on whole person activity rather than on thinking as separate from doing, implies a negation of the conventional division between mind and body; (iii) cognition is distributed across persons, activity and setting (Lave, 1988).

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Encoding treatments in geometry : an analysis of pupils'procedures[1]

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The work presented here is a part of an on-going qualitative study on the utilization of encoding treatments in the resolution of geometrical problems by junior high-school pupils.

Our previous studies (Mesquita, 1994, in press) suggest that the utilization of some non-standard external representations, in particular the ones in which we can not extract directly geometrical properties from, seems to foster pupils'high-level geometrical procedures and in particular encoding treatments.

In our communication, we will analyse and discuss the utilization of encoding treatments by 11-12 years-old pupils of two classes of a French junior high-school when they solved geometrical problems with this kind of representations; in particular, we will focus on the analysis of characteristics of their treatments. One characteristic is linked to the spontaneousness of these treatments, which are not a current practice in French school. Pupils used encoding marks, such as numerical ciphers, abstract marks or colours, in very personal manners. Encoding marks seem to be an effective contribution to write correctly a solution or at least to answer correctly to the question. Our communication will be elucidated by fragments of pupils' answers.

We argue that these personal procedures appear to be an emergence of symbolic treatments.

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LONG TERM EFFECTS OF A GEOMETRY COURSE BASED ON THE VAN HIELE THEORY

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In 1990, a teaching experiment was developed in Rio de Janeiro (Nasser, 1992a), involving five 7th grade classes from two state schools, aiming to: detect some causes of the difficulties presented by secondary school students in Geometry; identify the van Hiele levels (van Hiele, 1986) reached by the Brazilian students in the sample at that grade; investigate if the van Hiele theory can facilitate the geometry learning.

According to a quasi-experimental design, this research compared the learning of the topic of "congruence of plane shapes". While the control group followed the traditional approach of the textbook, the experimental classes were exposed to a teaching material, specially devised, based on the van Hiele theory. Since the students were reasoning at the two first van Hiele levels, the exercises requiring proofs and formal justifications had to be avoided at the beginning. The isometries (reflection, translation and rotation) were used, instead, to justify the congruence of two shapes.

The results of this research showed that the experimental students performed better than the control ones, mainly on questions requiring argumentation and justifications (Nasser, 1992b).

This investigation

In order to investigate if the special treatment caused any effect over the experimental students on that sample, the performance in Geometry of the students from one of the schools (Colégio Pedro II) has been monitored for the last four years. The data collected involved the marks obtained on common examinations given to all students at each grade, reports of these students' Mathematics teachers about their performances in class, and the results of the examinations to select students for the Universities in Rio de Janeiro, applied in the beginning of 1995.

The analysis of the data shows that the experimental students overperformed the control ones, taking active and expontaneous part on classroom work. These results give evidence of a positive long term effect of the experimental approach based on the van Hiele theory.

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STERNBERG AUTOMATIC PROCESSING INFORMATION AND KRUTETSKII ABILITY FOR THINKING OF CURTAILED STRUCTURE: A EXPLORATORY STUDY.

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Sternberg claimed that the ability for automatic processing information (API) is a key aspect of intelligence. In a similar way, Krutetskii pointed out that the ability for thinking of curtailed structure (TCS) is an important element of the mathematical abilities structures. Whereas both concepts have important similarities, it was investigated that if the API ability and the ability for TCS are two terms for the same concept or are two different phenomena. 69 university graduated students were administered (1) the verbal reasoning test of DAT, (2) a mathematical reasoning test based on Krutetskii theory, (3) three API abilities tests and (4) three TCS abilities tests. For the data analysis was used an Unweighted Leas Squares Factor Analysis and a Nested design MANOVA. The MANOVA dependent variables were the factors obtained in the Factor Analysis, and the grouping variables were (1) the type of graduate program (mathematical or verbal prevalence in the students graduate programs) and (2) the academic achievement (below and above the median into each type of graduate programs). The results are concordance with the claim that the API ability and the ability for TCS are different phenomena. Theoretical, practical and methodological implications are discussed.

Kuhn Revisited

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Four concepts essentially determine the architecture of Kuhn's essay, The Structure of Scientific Revolutions: Paradigm, Normal Science, Crisis, and Scientific Revolution. Kuhn's fundamental concepts may be grouped into two complexes. On one side, we have paradigms and normal science, on the other, crises and revolutions. The tension between these two conceptual complexes has been observed early. Scheffler, for instance, claimed that they are cognitively incompatible or incongruous with one another (Scheffler, 1967, 1968). I think that Scheffler did not sufficiently take into account the active and instrumental character of scientific cognition (Otte, 1990). Nobody is able to get cognitively involved with more than one theoretical perspective at a time and still maintain the promise of a successful application. It may take a whole individual life to work out the consequences of a particular perspective. If we take this into account and consider the matter strictly within a context of communication, what has been outlined in Scheffler's critique becomes more interesting, in particular now that the interrelationship between science and society has attained new dimensions with the influence of complex scientific technologies in all walks of life (society as a laboratory). It may be too late when finally the consequences of a large scale "experiment", of the implementation of a radically new, research based innovation become tangible.

Kuhn's work is considered to have achieved a revolution both in the philosophy and in the historiography of science a self-evident result of this is that his story has to be told twice. First, more or less immanently, so-to-say as a treatise in the history of epistemology or philosophy of science which is designated on grounds of certain hypotheses about the evolution of modern western rationality. Second, however, the changes in philosophy of science and historiography of science introduced by Kuhn can also be seen as a case of self-application of the message of his own essay. One could then ask what genuinely singular historical context has produced the revolution in the views on science which Kuhn's book puts forward. Process of change in the conceptions of science as they emerged after World War II are responsible.

INVESTIGATING THE IMPACT OF CHANGE IN THE CALCULUS CURRICULUM AT SAN JOSE STATE UNIVERSITY

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The Department of Mathematics and Computer Science at San José State University (SJSU), a large college with over 26,000 students decided to change the curriculum of the first calculus course from a traditional to reform course. This departmental decision to change the curriculum encompassed content, technology and pedagogical modifications. The content moved from the traditional to exploration based curriculum using the Harvard text, beginning with a library of functions and introducing both derivatives and integrals through student experimentation, approximations and observations. Technology became an assumed tool for student experimentation. The role of technology included visualization, experimentation, verification of conjectures, numerical calculations, and graphical representations. Class environment changed to support more student participation and group work. Along with these changes came a commitment on behalf of the faculty to investigate the impact of this change. Study of the changes extended beyond that of the cumulative score on a common final. Rather, it included an exploration of students attitudes, beliefs and understandings.

The purpose of this paper is to share interim results of these efforts now underway at SJSU. To investigate the impact of the curriculum change in the first semester of the calculus curriculum, results from the pretest, middle of the semester test, the final exam, student grades and initial survey result from the subsequent course will be reviewed. Although the data covers 10 sections of calculus and includes 4 different surveys, the focus of this report will be on the understanding of key concepts, the concept images (Tall, 1992) and the interaction between the understandings and the affective responses. The review will concentrate on a small subset of students and underscore the linkages between beliefs and understandings. Interactions between the curriculum, student beliefs and student concept images will be developed in light of work reported in Kaput and Dubinsky (1994) and theoretical frameworks suggested by Dubinsky (1992) and Thompson (1994).

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STEPS TOWARDS EFFECTIVE CLASSROOM PRACTICE

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One important part of the preparation of a mathematics class is the selection of tasks to be tacked by students. However, a given task can be transformed in mathematics activity in a number of quite different ways, depending on the teacher interpretation of what is important, her perception of students' abilities and interests, the time available, etc. This presentation discusses how a mathematics teacher, Marta, concerned with the implementation of a new mathematics curriculum stressing the use of materials and pupil activity, leads a 8th grade class in the concept of area.

Marta is about 35 years old, having some ten years of experience. She has a positive attitude regarding the new curriculum and meets regularly with two colleagues from her school to design extensive and carefully crafted worksheets, containing different kinds of mathematical tasks, first for 7th grade and now for 8th grade. These worksheets take a structuring role in each teaching unit. For a given class, she either uses tasks taken from this worksheet or supplements it with extra tasks.

In this way, this teacher participates of a regular practice of informal discussion and production of materials with other colleagues. She is quite enthusiastic about this work, that she finds the most significant professional experience since student-teaching.

As many other mathematics teachers, she likes the idea of "doing things different" in the classroom, but she is aware of the need to cover all the topics of the national curriculum. She solves this issue in a positive way, jumping over some of the curriculum topics in exchange for a better exploration of others.

However, the prevalence of a major concern with "basic skills", such as numerical computation and knowledge of mathematical terminology as well as some decisions that she takes about classroom management (such as teacher and student roles an use of the time) strongly affects the scope of the innovative tasks that she takes to the classroom.

Willingness to innovate and openness to new ideas is just a first step towards an effective classroom practice. The teacher needs to be able to analyze and reflect about what is going on in the classroom, specially in what concerns the nature of the teaching situations and the activity undertaken by students — and mathematics education research should be able to provide conceptual and practical means to make it happen.

STUDENTS'APPROACHES TO DIFFERENT USES OF VARIABLE.

RICARDO QUINTERO & SONIA URSINI, CINVESTAV
ARACELI REYES & MARÍA TRIGUEROS, ITAM
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Aims of the study: Our research project is concerned with the mechanisms students of different school levels use to identify, discriminate and connect different uses of variable (specific unknown, general number, functionally related variables) and how these mechanisms are affected by instruction. With this focus in mind, we intend: 1) To elaborate a profile of students' interpretation and symbolization of different realizations of variable at different school levels: pre-algebraic pupils (12-13 years old); algebra beginners (15-16 years old); beginning college students (18-20 years old). 2) To investigate to what extent the difficulties students have to cope with different uses of variable depend on the cognitive demands of the concepts themselves, or on specific features of school instruction. 3) To investigate the usefulness of working in structured computer-based environments, to help students in discriminating and connecting the multifarious aspects of variable.

For dealing with 1) a questionnaire of 52 open ended items was developed, extending a kernel of questions used by Ursini (1994) who studied pre-algebraic pupils' interpretations and symbolizations of variable. It was applied to 73 beginning college students in Mexico City. The answers were analized both, qualitatively and using Clasical Test Theory (CTT).

Preliminary conclusions: It stands out that the concept of variable is not firmly established among college students. They do better with variable as specific unknown than with the other aspects although they still have difficulties with this use of variable. It is in the handling of this characterization where differences with pre-algebraic students were more evident, which may be due to school influence. It is remarkable that college students have great difficulties with the other two characterization of variable and that a tendency remains to interpret them as specific unknown, being so the difference with pre-algebraic students small. These preliminary results stress that there is not enough emphasis on these aspects of variable at school. The variable concept it favours is fragmented and this could explain students defficiencies whithin algebra itself and with the learning of other branches of mathematics as analytic geometry, calculus or statistics that require a fluid handling of variable.

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COGNITIVE STYLES, LEARNING and TEACHING MATHEMATICS

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Type of Research : n°41-42

The aim of this communication is to present an "action-research" like study conducted in France by a team of about ten Mathematics teachers from three Lycées (high schools) in Burgundy on the "cognitive styles" of pupils in a classroom situation. The pupils concerned are first, second and third year pupils. Most of them are 16-18 years old.

The study originated, first in the difficulties encoutered by teachers in teaching their pupils Mathematics, and secondly in the introduction of a new organisation in the schoolwork : the creation of "modules" in the pupils' timetable aiming at offering them individual help in their learning Mathematics.

The relection and work on the notions of "cognitive style" and "cognitive profile", of "learning style" and "teaching style", of "educatinal differenciation" seemed to them attractive prospective fields, likely to offer pragmatic answers to their educational difficulties. This approach must be considered more in a praxeological way then in a "fundamentalist" one. The notion of "cognitive style" developed by psychology must not be given a theoretical basis but must rather be used as a factor whose knowledge by the teacher and the learner is likely to increase the efficiency of the didactic sequences.

This prospect has led the towards some problematics of praxeological nature based on the following main questions :

Presupposing the existence of "cognitive styles" among individuals,

- how can be the teacher clarify the "cognitive profile" of the learner in the complex context of the classroom ?

- what links are there between the "cognitive profile" of a learner and his attitudes in learning Mathematics ?

- how can the didactic sequences be built, bearing in mind the double constraint of the teaching style linked to the "cognitive profile" of the teacher-subject and each learning style linked to the "cognitive profile" of rhe each learner-subject ?

- how can the teacher help the learner capitalise on this knowledge of his own "personality" in the learning process ?

The communication will present the pragmatic answers provided by the group whose teaching pratice involves differenciation, and their assessment methods. A thirty-page detailed article will be to anybody interested.

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A DEVELOPMENTAL STUDY OF TIME AS INTEGRATION OF DISTANCE AND VELOCITY

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This study follows on from Wilkening's (1981) study and aims to verify whether children and adults are able to judge time using information about speed and distance. In the experimental task the value of two dimensions, speed and distance, were given and subjects had to infer the value of a third dimension, time. The sample consisted of 45 subjects from middle class backgrounds: 15 university students, 15 10-year-olds and 15 5-year-olds. All subjects were tested under two conditions: manual and verbal. The E explained that three animals (a turtle, a rooster or hen, and a cat) liked to play near a dog's house, but they were afraid of the dog. When the dog began to bark, they ran away as quickly as they could across a bridge located in front of the house and they stopped immediately when the dog became silent. In the manual condition the subject was shown a button placed in front of him or her, and it was explained that upon pressing the button the dog would begin to bark and would cease barking only when the subject pressed the button again. Each animal was then placed at a pre-determined point and the subject was asked to indicate how long the dog would bark before the animal had moved a certain distance (70 cm, 140 cm, or 210 cm). The subject was asked to press the button to make the dog begin barking and to press again when the subject thought the dog had barked long enough for the animal to reach that point. When a verbal response was requested, the subject instead of pressing a button when the animal was deemed to have reached the given point, told the researcher to stop the barking. Results indicate that integration rules depend upon age. The integration concept was found only for adults and 10-year-old children. While the two older age groups obeyed the normative division rule, the 5-year-olds shifted to a simpler subtraction rule. These results confirm Wilkening's main findings with a German sample

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UNDERSTANDING SHAPES AND SPACE FROM REFLECTION AND ROTATION

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Piaget[1] described geometry mainly as a science of space. He made extensive studies of children's logical thinking and of geometrical concepts with implications for the teaching geometry. He described the process of children thinking and a method for investigation of an individual is thinking and understanding. Van Hiele's[2] theory uses a geometric model as a methodology that allows students to move through a sequence of levels of geometric development. The significant contribution of the Van Hiele and Piaget studies to research, has been in the dynamic development of teaching-learning methodologies. According to Piaget[3] the manipulation of concrete objects forms the basis of people's knowledge. The activities that involve manipulative material are spontaneous and are essential for children to attain experience in spatial perception. Fuys[4] et al. reported that manipulative materials are essential aids in learning geometry, specially for students at lower levels in Van Hiele hierarchy.

This study provides a description of children's reasoning in reflections and rotations in terms of their answers to a test in this subject[5].

27 English children, aged 12 and 13, at 8 year of their schooling were tested. 13 of them did not use concrete material to answer the test, and 14 children used concrete material to find the images of the objects given in the test. The task involved drawing the reflection and rotation of given objects, children were asked to find centres of rotation and they also decided if a point plotted is or is not the centre of rotation.

The results of the test, indicate that pupils found it easier to answer the questions about rotation using the concrete material than without it. Results for the questions on reflection were less clear cut.. The detailed results and statistics tests will be shown in the conference.

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MATHEMATICS LEARNING AS SITUATED LEARNING*

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The school is a *site* of cultural transmission where, teachers and students' are involved in *everyday activities*, in the sense that: (a) they have routine character, (b) they rouse expectations generated over times about its shape and settings, and (c) their living is in settings designed for those activities and organized by them (Lave, 1988).

We believe that learning is situated in practice as "an integral part of generative social practice in the lived-in world" (Lave, 1991, p. 35). This perspective is consonant with Vygotsky approach that "does not separate individuals from the sociocultural setting in which they function" (Wertsch, 1985, p. 16).

We (a) look at mathematics as "an act of giving sense which is socially transmited and constructed" (Schoenfeld, 1992, p.339); and (b) talk about mathematics learning as "learning to think mathematically". We also think of mathematics as "a kind of action whose meaning is not determined by the fact that it is *math*, but by its place in a sociocultural system of activity — be schooling, the household, or an occupation" (Lave, 1992, p. 84).

It is in this context that we are trying to understand school mathematical learning, observing and analyzing a small group of 8th grade Portuguese students in their mathematics classroom. We considered the unit of analysis proposed by Lave (1988) as "(...) the activity of persons-acting in setting" (p. 177).

We will present and discuss some preliminary results such as: (1) pupils' individual goals (to be ahead, to understand what I am doing, to do what the teacher asked to do, to be accepted by the more *powerful* one) show how pupils are participating in different activity settings; (2) elements that help pupils to give meaning to the activity are identified (the existence of a partner who is a good listener, materials they choose to use, teacher questions or suggestions).

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Practical knowledge vs. mathematical knowledge : an example from the functions

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In this presentation, we attempt to study the process by which the knowledge that students use to face familiar problems is transformed into formal mathematical knowledge. This process, so called "institutionalization", is examined in real classroom conditions with a sample of 13 years old students.

At the first stage of the study we gave the students a series of specially chosen familiar problems in order to introduce them to the concept of function. After several successful sessions the students were able to make the necessary generalization and the "institutionalization" of the concepts involved. Then, we gave them several mathematical problems related to the previous ones. By comparing the performance of the sample on familiar problems with their performance on mathematical problems we found that the subjects of the study faced difficulties in applying the generalizations of the solutions used previously.

The conclusion of the study deals with the ways that the students of this age group use to relate their practical knowledge with the formal one and the concepts they entertain in relation to the different status of these two kinds of knowledge.

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PROBLEM SOLVING: PERFORMANCES OF PRESERVICE TEACHERS

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It is generally accepted that teachers' knowledges, beliefs and attitudes towards mathematics and mathematical problem solving, seem to play a determinant role in the actions they undertake in the classroom and their interactions with the students. So, research efforts must be directed to understand what teachers know, believe and feel about mathematical problem solving and its teaching (Brown,1993; Pajares,1992). Otherwise, my personal experience tells me that mathematical problem solving is one of the most neglected subjects in our schools.

Based on this same assumption and as a mathematics teacher educator, the underlying idea of this research has to do with the need to get a better understanding of preservice teacher's knowledge, attitudes, performances and to reflect on the problem solving programme I propose to my students. So, this research involved a class of the 4th year of the Maths and Science course in a School of Higher Education, for almost five months. I studied in this class, the performances and the reactions of these future young teachers when they were involved in problem solving tasks. In particular, I sought to know more about the proposed tasks, as well as to identify their main difficulties and attitudes. The following are the fundamental questions: (1) What patterns are revealed by the participants in the written problem solving tasks?; (2) What difficulties are revealed by the participants in the written problem solving tasks; (3) How could we characterize the attitudes of the participants while they carry out their tasks? (4) How could we characterize the performance of these future teachers in this activity, after formal instruction in mathematical problem solving?

Taking into account the nature of the overall purpose of this research, I decided to focus on a qualitative methodology, which could provide the grounds for an understanding of the performance in problem solving of a class of young teachers and their relationship with other factors. In the process of data collection some data sources were used: four problems, four performance inventories, one report and one attitude task. The data analysis was holistic, descriptive and interpretative. Repeated analyses of all data collected and the research questions were done to enable me to come up with categories. These categories were used to organize all data collected.

The results showed there was a middle grade of performance, in particular in two of the problems. The attitudes of the participants, in general, it was positive. They enjoyed the proposed tasks, which they found interesting and appropriate for their future students. The patterns used were those I expected for the kind of problems proposed. The difficulties were mainly in the comprehension of the problem, generalization and taking decisions. They were not reflectives ones. They gave little significance to the phases of the Polya model, were organized in the written work and they found the strategies useful.

A THEORY OF MEASURE

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In seeking to determine the obstacles to the understanding of numbers and the operations on numbers, it was recognized that the numbers of arithmetic were the numerical components of quantities which arise in measurement situations. An analysis of the context of a variety of measurement situations resulted in a number of fundamental properties common to all. A *measurement situation* consists of a *domain*, $\mathcal{M}$ of objects and a *measure*, a property common to all the objects. There is a procedure for determining when two objects have the same measure. This defines the measure *operationally* and defines an equivalence relation in the domain and separates the domain into equivalence or *measure* classes. These measure classes are objects in the ***measure system*** $|\mathcal{M}|$.

There are procedures for *dissecting* an object into subobjects and *consolidating* objects into another object. (For Cuisenaire Rods, two rods have the same length if one fits exactly on top of the other. A two car train equivalent to a rod is a dissection of that rod and a rod equivalent to a two car train is a consolidation of that rod. The measure classes are designated by color.) Dissection and consolidation are consistent with the equivalence relation and induce an operation on the measure classes which is addition. The measure system with this addition is a semi-group.

Repeated dissection of the subjects of an object produces a *set* of subobjects, a *partition* of the object. Of special importance are U-partitions where all the subobjects in the partition are equivalent to an object U, the *unit.* The subset of U-partitionable objects $\mathcal{M}_U$ is a sub-measurement situation of $\mathcal{M}$. Partitions, as sets are objects in the measurement situation of sets, $\mathcal{S}$ where the measure is numerosity. This induces a mapping from $|\mathcal{M}|$ to $|\mathcal{S}|$. Counting defines a mapping from $\mathcal{S}$ to $\mathcal{N}$ the set of numbers, where numbers are defined as sequences of numerlogs. (The sequence <one, two, three, four, five> is the number 5.) Equivalent sets have the same count so that the mapping induces a mapping from the measure system of sets, $|\mathcal{S}|$ to numbers $\mathcal{N}$. Composing this map with the map from $|\mathcal{M}_U|$ to $|\mathcal{S}|$ yields a map from $|\mathcal{M}_U|$ to $\mathcal{N}$ which assigns to an object in $\mathcal{M}$ a quantity with unit U.

The addition of two numbers a and b in N is defined by first finding two objects in M with quantities aU and bU and adding their measures. The sum of these measures is given as a quantity with unit U by cU and we define $a + b = c$.

Changing units from U to V where U is V-partitionable leads to multiplication and fractions. Extending quantities to $\mathcal{M}$ from $\mathcal{M}_U$ leads irrational numbers, rounding, accuracy and approximation.

POSTERS

EFFECTS OF TRAINING-IN-SERVICE ON TEACHERS' REPRESENTATIONS ABOUT TEACHING AND LEARNING INTEGERS

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The present study investigated 25 high school teachers' conceptions about relative numbers and also the effects of analyzing sessions of didactic activities presented during a period of training-in-service on these teachers' representations about teaching in this numerical field, in particular, the multiplication operation. The answers given in different moments of the period of training were analyzed. This training consisted of the presentation and analysis of didactic situations that aim to justify that when multiplying integers the product is positive when the two factors are negative. During this process these activities were also analyzed by the systemic perspective of Mathematics Education (Henry, 1991), in particular by the aspects related to the didactic transposition (Chevallard, 1985). These activities, designed based on several studies (Freudenthal, 1973; Glaeser, 1985 and Kobayashi, 1988), explored different situations: **measuring volume during a certain period of time and registration of profits and losses during a certain period of time, inductive extrapolation on numbers and linear function graphs and geometrical representation of algebraic expression.** The main obstacle indicated by teachers on teaching integers refers to their own comprehension of the nature of this concept, and they consider the use of the sign rule the greatest difficulty presented by students. In the beginning of the training period, 20 teachers said that they used a rule, or showed a lack of comprehension, to justify why (-) x (-) = (+). Only five said they used a mnemonic artifice or some graphical representation. Apparently occurred a change of perspective, during the training process, from a teaching using mere memorization of rules to another one with comprehension and utilization of significant situations by the students.

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MATHEMATICAL PROJECTS IN THE EDUCATIONAL SYSTEM IN ISRAEL - DEVELOPMENT, CHARACTERISTICS AND EVALUATION

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The educational system in Israel appears to be centralized from the point of view of resources and curriculum. These are determined by the Ministry of Education with the cooperation of academic and field experts. In spite of this centralization, there is a large amount of flexibility in the application of the curriculum, in the choice of instruction methodology and, specifically, in the evaluation methods. In all stages of elementary, middle and secondary school, assessment is done by teachers, without systematic external intervention. Only at the end of secondary school are there external matriculation exams. The teachers are involved in their preparation and grading; the final high school grade is the average of the external examination and teacher assessment of class work.

The fact that there is no required external assessment allows diversity in assessment methods during different stages of the educational process. One of the innovative tools being applied more and more are mathematics projects used for enrichment and deepening of the educational process, and for assessment. These projects are carried out in all stages of the learning process, although they differ in their characteristics, development and assessment method, according to age and scope of learning.

In elementary school, the mathematics project is generally a review of a variety of mathematics subjects that were previously learned. Emphasis is placed on interdisciplinarity: students may plan a neighborhood based on geographic characteristics, by applying mathematical tools, or carry out small research projects in the natural sciences using mathematical models. Advising and assessment is carried out by the school teachers.

In high school the projects are mainly extra-curricular. The students must cope with subjects that are not included in the school curriculum and that require dealing with new mathematical knowledge or with a combination of mathematics and other disciplines. Because of the need for specific expertise, guidance and assessment are done by the teacher in cooperation with academic faculty.

The use of mathematics projects in the learning process has extremely positive implications for motivation and interest shown by a wide cross-section of mathematics students. In addition, the special criteria according to which the projects are assessed have implications on the evaluation methods used in mathematics and other subjects as well.

In the poster we will present the development, control and assessment methods used in the projects. In addition, we will present examples of projects carried out in Israel.

GRADE DISTRIBUTION AND STABILITY OF ATTITUDES TOWARD MATHEMATICS

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The present paper is part of a wider research concerning the attitudes toward Mathematics realized in four schools of the region of Campinas. The data was obtained from the answers of 2007 students to a questionnaire and to an Attitude Inventory (Aiken, 1979). The Attitude Inventory was previously traduced and adapted to Brazilian children and the statistic analysis results showed a satisfactory Reliability Level (0.96). As the purpose was to improve the statistical treatment, in this paper the obtained data of 1944 subjects was used and distributed according the following table:

DISTRIBUTION BY GRADE

Value Label	Frequency	Percent	Valid Percent	Cum. Percent
Grade 3-4	541	27.8	27.8	27.8
Grade 5-6	466	24.0	24.0	51.8
Grade 7-8	302	15.5	15.5	67.3
Grade 9-11	635	32.7	32.7	100.0
Total	1944	100.0	100.0	

There are only few school situation studies realized in Brazil regarding attitudes related to Mathematics and within these, no one refers to Mathematics in itself, but only toward component aspects of Mathematics teaching, i. e. attitudes related to the solution of problems (Lindgren and others, 1964) and some monographs related to the teacher, the teaching method, or both. The international literature presents many studies regarding attitudes related to Mathematics, particularly in the U.S.A., and periodical revisions of these studies are being done (Aiken, 1970), besides the studies involving comparatively various countries (Walberg, Harnish and Tsai, 1986). Concerning the relation between the series and the attitudes, Haladyna and Thomas (1977), having as subjects 2800 students of first level, verified that the attitudes of these students related to Mathematics change almost imperceptibly in the initial series (1. to 6. series) but decline abruptly in Grades 5, 6 and 8. Other authors found decline in preference for Mathematics from Grade 5 on. Martin et colab. (1991), using different measuring instruments, observed the development of negative feelings related to Mathematics as the students progress in school, the fourth and the seventh series being the ones where students demonstrate the most negative attitudes.

The analysis of data (Duncan Procedure) denotes pair of groups significantly different at the .050 level. The results of the present paper permits us to conclude, at least for this sample, that attitudes toward Mathematics do not modify themselves in a continuous and systematic manner as the student progresses in the series. In this study, attitudes are more negative in students of Grade 7 and 8 and this can be attributed to the exigencies of abstraction towards these students, who, up to now, were working much more with arithmetical concepts and, from Grade 5 on, are solicited to work with algebraic contents. Besides, we have to consider the existing differences between the first four series and the four last ones, in relation to the teaching method, the teacher's formation, the quantity of subjects, etc.

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DEVELOPMENT OF EQUALITY REPRESENTATION IN SIX TO THIRTEEN YEAR-OLD CHILDREN

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The passage from arithmetic to algebra has been a topic of interest in the field of psychology of mathematics education. Many researchers have tried to clarify the most important epistemological obstacles in this passage (e.g., Filloy and Rojano, 1984), as well as proposing ways to face them. Among the above mentioned obstacles, one of the most important in the conceptual field (Vergnaud, 1991) of algebra is the representational transposition from natural language (in which word problems are expressed) to algebraic-formal representation (Laborde, 1982; Garançon, Kieran e Boileau, 1990; Capponi, 1990; Da Rocha Falcão, 1992). In a previous study led by Schliemann, Brito Lima e Lins Lessa (1994), with 6 to 11 year-old children, it was described some spontaneous representational procedures concerning the passage from problems in natural language to the algebraic equation; this study showed, in a non-systematic way, the longitudinal development of children's algebraic representations. Bodanskii (1990), on the same issue, proposes some pedagogical initiatives aiming to develop some representational skills (e.g., detection of basic relations expressed in the problem) in children as young as 6 years-old.

This study shares this interest in algebraic representation, and aimed to contribute to the comprehension of its development through the analysis of 72 children, with ages varying from 6 to 13 years (six groups of 12 subjects each). All children were invited to solve 12 algebraic problems, presented as short stories during an unique session conducted as a clinical interview; each problem was presented in two versions: stressing transformation of a quantity into another, or stressing the equality between two quantities.

Results immediately available show that written representation for algebraic problems does not appear spontaneously, being only detected in age-levels corresponding to the moment of introduction to algebra at school (6^{a} grade in Brazilian school system); at this age-level, certain subjects can for the first time represent a problem in terms of an equation. It was also detected that problems mentioning *half* were consistently more difficult to represent than those mentioning *double* and *triple*. Problems with unknowns in both sides of the equation were the most difficult for the children to solve; 10-11 year-old subjects (5^{th} grade) have only used arithmetic trial-and-error procedures, substituting tentatively unknowns by numbers. This problem-solving procedure, by the way, was more and more used with the diminution of school level; the youngest subjects (1^{st} grade) frequently proposed one of the values appearing in the problem in order to fill in the unknown. As for the variable *version of the problem* (equality versus transformation), problems stressing the equality between quantities seem in general terms to facilitate the representation and manipulation of resolution when it is necessary to operate over the equation in order to reach the solution. Other aspects, concerning a more clinical analysis of protocols production, combined with a statistical multidimensional analysis, will certainly complete and clarify the aspects already available.

NEW APPROACHES TO NEUROSCIENTIFIC RESEARCH AND THEIR IMPLICATIONS FOR MATHEMATICS EDUCATION

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The prominent place of symbol manipulation, oral and written communication in schools may have stemmed from the early studies of brain activity, especially by Broca and Wernicke which focused on language as a mechanism of thinking. The belief that learning should consist of repeated practice can be derived from Hebb's (1949) work on long-term potentiation.

Prior to the late 1970's research into higher cortical function was limited because effective, nonevasive techniques were unavailable. Recently, magnetic resonance imaging (MRI), positron emission tomography (PET) and single-photon emission computerized tomography (SPECT) have made it possible to study cognitive functioning under more normal conditions. Recent research on the role of nitric oxide in brain functioning suggests that cells in close proximity communicate much more rapidly that synaptic theory would predict. Such results help us understand how the brain uses images in such powerful ways.

As mentioned previously the areas of the brain which are specialized for the understanding and production of language are small and relatively discrete. In contrast, the areas devoted to the processing of visual information are relatively large and spread over wide areas of the cortex. This fact has been interpreted to mean that visual processing is actually much more important that the processing of speech (Kosslyn, 1994). Certainly, our research and that of other mathematics educators (Brown & Wheatley, in press) suggests that visual imagery is important in mathematics learning. For this reason we have chosen to focus on research concerning spatial reasoning and visual imagery.

Recent research on cortical functioning suggests that planning mathematics lessons which emphasize sequential processing may not fit well with how our brains work. The emphasis on practice in mathematics learning which seemed necessary using the Hebb model, comes into serious question. Imagery may play a much more important role in mathematical activity than previously thought. This research suggests that in mathematics education we can gain much by drawing on new models of brain functioning.

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Situated Generalization

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Learning psychologists have traditionally discussed how previously acquired knowledge is used in new situations in the context of transfer. Transfer entails the extension of knowledge to conditions broader than those under which it was initially acquired and, as such is also closely associated with the process of generalization, i.e., the extent to which the individual can articulate about similarities across settings or formulate general statements about hypothesized regularities.

Results from decades of research on transfer within laboratories have been notoriously frustrating: Studies documenting transfer are quite rare; studies documenting failure to transfer abound. More recently, "situated learning" approaches have confronted theoretical and methodological suppositions of transfer studies arguing that learning and thinking take place in specific contexts that, far from being incidental, are essential to what is learned and thought. As much as these developments represent progress, conceptions of situated learning have not provided us with an account of how ideas are generalized beyond the idiosyncrasies of their originating contexts, or how, despite specific differences between two situations, one can identify essential similarities between them or try to approach both in the same fashion. They even appear to contradict the very idea of generalization, for it is tempting to believe that the more learning is bound to particular situations, the less one will be able to recognize similarities and common patterns.

The aim of this paper is to show that the ways in which students learn to deal with new specific situations involves remarkable use of previous knowledge as analogies, categorizations, comparisons across situations, search for correspondences between different settings, as well as generalizations, and that to recognize them as such one needs to set aside stereotyped/formal views about what transfer and use of previous knowledge are.

Our view is that if we try to analyze transfer of knowledge by pre-determining the common features or structures between a starting and a target situation, expecting that learners will focus upon these while learning about the first task and that, latter, they will apply it to solve the second task, we are bound to fall in finding many occasions where transfer does occur. If, however, we look at the learner own ways for making sense of new situations we may find that previous knowledge is constantly used.

We illustrate our views with two cases of children trying to make sense of graphs. Both children spent four or five one hour sessions working with an interviewer about the interpretation of different symbolic systems for representing "trips". By examining instances from their video-taped interviews, we can obtain glimpses of how previous knowledge and understanding are summoned to the problems at hand. Each child was introduced to a learning environment in which real-time graphs of distance over time were automatically produced from their own movements or from the movements of a train moved under their control, by means of a motion detector linked to a computer. The situation is one where events in the room can be related to the shape of the graph on the screen. Our analysis focus not only upon use of knowledge developed before the series of interviews started, but also at how the learners try to relate the different events they are trying to cope with during the interview.

Analysis of interview videos revealed, in both cases, constant use of previous knowledge to make sense of the new situations. But this use was not a formal application of what was already known, but rather a dynamic and open ended way of relating previous experiences to new ones. In this process, not only the understanding of the new situation was enriched by previous knowledge, but also previous ideas were transformed, expanded, and enriched by new experiences. We believe that, through such type of analysis, it is possible to reconcile the apparent contradiction between situated learning and the development of generalizations.

In what they believe?

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Key words: Beliefs, beginning teacher.

Abstract: The majority of authors agree on the importance of the teacher as a key figure in the classroom. In a mathematics' classroom this is particularly true because the nature of mathematical knowledge is itself problematic (Ponte, 1994) and it's urgent, according to the NCTM (1989), to begin to learn to valuate mathematics and to be confident about own's abilities.

The growing importance of the teacher as a central figure is obvious on the quantity of research done. Specially, since the work of Thompson (1982) and Cooney (1985) standing on the assumption that what the teacher does in the classroom reflects his beliefs. Many of the studies are about pre or in service teachers. In our work, we are in interested in the beliefs of the beginning teacher.

In the present poster we look at some beliefs about mathematics, mathematics students and teaching mathematics. The subjects (N=53) were invited to write an imaginary dialogue between three characters: mathematics, a mathematic's teacher and a mathematic's student. The presentation displays some of the results obtained in this study.

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VALUE AND PRICE ESTIMATION BASED ON SOCIO-CULTURAL CUES IN ADULTS OF DIFFERENT SCHOOL LEVELS

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Brazilian people has been submitted to a very unstable economical system, were high rates of inflation and successive changes in currency make ordinary people loose reference in the evaluation of what is cheap or expensive. Because of this specific difficulty, we can observe many cultural procedures of price estimation based on alternative value references, in order to get rid of the senseless monetary reference: in Recife, for example, we have observed in private (familiar) contexts some articles having their prices established in units of *bottles of beer*. Nowadays, with the last Brazilian currency, the ***real***, prices are much more stable, but the cultural difficulty in price estimation and evaluation is always present.

Three aspects seems to be relevant in the analysis of this cultural activity of price estimation in a context of a economy traditionally unstable: 1. Comprehension of the decimal system upon which most currencies are built up, as well as some mathematical operations involving basic arithmetic and proportionality; 2.Level of socio-cultural familiarity with this kind of activity (establishment of price/value relationship), as shown by Saxe (1993, 1991) in this study of Brazilian candy sellers at Recife; 3. Cultural system of price hierarchy (*how many bicycles does a Volkswagen worth?*).

This study aimed to describe more accurately the usual strategies involving price estimation, comparison and proposition in relation with mathematical structures involved (different levels of difficulty in the context of the multiplicative structures), level of socio-cultural familiarity with price system (people professionally involved in commercial activities versus other groups) and school level (very elementary level versus high-school or equivalent).

Results interestingly show no differences among the groups concerning problems involving the most elementary problems of proportionality, all groups made frequent use of auxiliary cultural references in price estimation, official Brazilian minimum wage (R$ 70,00 = US$ 61,60 by the time of data collecting) appearing as the most frequent unit of reference for comparisons and decisions in terms of the price [cheap or expensive] of proposed products. More detailed clinical analyses were initiated in order to clarify certain differences between groups in terms of problem-solving procedures.

Playing with multiples and arithmetic patterns in a domino game

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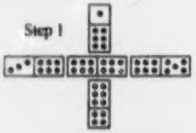
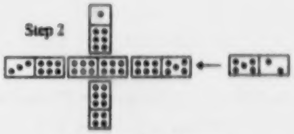
Many studies about the relations between culture and cognition have focused on economic activities as an important setting for mathematical learning (Nunes, Schliemann & Carraher, 1993), while few have investigated recreation practices as a source of mathematical knowledge (Saxe & Bermudez, 1992). The research reported in this poster aims at studying the practice of game playing as an unique setting for learning mathematics. A community of poorly schooled adults living in an agricultural area of inner Brazil, developed over many years an specially interesting variation of the traditional domino. The goal of their domino game is to produce arrangements of the pieces such that the sum of values in all ends (four, instead of the usual two ends) is a multiple of five (see diagram below). Several rounds of this game is played until one of four contenders reach 200 points.



3+1+5+4=13 ⇒ no points



13-5=8; 8+2=10 ⇒ 10 points

Several strategies are employed by the players, who need (1) make additions and subtractions for each piece they put in (see step 2 above); (2) think about the pieces an adversary might play; and (3) plan in advance which pieces can give the highest multiples of five.

Ethnographic observations of the games played in natural situations, problem solving sessions with arranged games (e.g., with bases other than five), and paper and pencil tasks were used to investigate (1) the players' understanding of multiples and of number sequences with a constant rate, and (2) the strategies developed with increasing expertise in the game. A preliminary analysis of this data suggests the uniqueness of this situation for the analysis of the acquisition of a mathematical concept in a recreation setting outside school. The poster will characterize the strategies used and their development with practice, showing at the same time that transfer of mathematical abilities developed in the game suffer important modifications when required in paper and pencil tasks. Finally, we plan to show how games which are familiar to schooled urban children may be used to enhance their understanding of specific mathematical concepts.

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REPRESENTATION OF THE LINK BETWEEN FREQUENCY AND PROBABILITY ON TEENAGERS

Comparison between french and brazilian pupils

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PIAGET, INHELDER [1], then FISCHBEIN [2] studied the impact of the ideas of hazard and probability on child and teenager behaviours. MAURY [3] and BORDIER [4] have developped their works on the learning of probability in scholastic context, linked with spontaneous conceptions that could be observed.

Particulary in France, the teaching of probabilities goes more and more towards an experimental approach of stabilisation phenomenons of the observed frequencies [5].

Such frequentist introduction of that probability concept to teenagers conflicts with some preconceived conceptions that must be focused on, in order to propose suited didactical engineerings.

We carried out a questionnaire [6] on the link between frequency and probability. That survey took place in France on pupils of 16 to 18 and in Brazil on university first year students. It resulted that the same wrong ideas were expressed by both french and brazilian teenagers.

As a consequence, no significant difference on that point could be explained by different elements such as language, culture, or the way of life. On the contrary, we observed the same epistemological obstacles that could be studied more precisely in the future.

The poster presents two questions and exemplary replies extracted from the questionnaire whitch complete version will be available. Statistics diagrams will illustrate them.

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A SURVEY STUDY OF BRAZILIAN STUDENTS' DIFFICULTIES IN THE LEARNING OF ELEMENTARY ALGEBRA

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The introduction to algebra represents for most students a moment of difficulty, since the passage from arithmetic to this new conceptual field is not a continuous, linear process. In fact, as many researchers have shown, there is an *epistemological gap* between these domains (Cortes, Vergnaud and Kavafian, 1990; Filloy and Rojano, 1984), whose consideration helps to explain some cognitive obstacles in the origin of several systematic mistakes. These obstacles are present in two main aspects of algebraic activity: the *proposition* of an equation (passage from natural language or empirical data to an algebraic notation) and the *resolution* of an equation. As shown by Laborde (1982) and Da Rocha Falcão (1992), the first of these two aspects seems to be specially important to take into account in the proposition of a didactic sequence in elementary algebra.

This study aimed to add data from a Brazilian north-eastern sample of 13 to 17 year-old students to the set of available data from other countries. A total of 386 protocols of students from the private school system of three north-eastern states of the Brazilian federation (Paraíba, Pernambuco and Alagoas) was collected with the help of their respective mathematics teachers, in the context of the usual collective class activity. The students were asked to solve 20 questions, corresponding to ten algebraic structures, each structure being presented as a word problem and an equation in the context of the same protocol.

Results clearly show a greater incidence of difficulties in word problems, where it was first necessary to propose an equation; in this case, the most frequent mistake was exactly due to the proposition of an incorrect equation to the problem; in the case of the equations, the most frequent mistake was caused by the incorrect utilisation of algebraic algorithms of simplification, specially those related to numerical coefficients and systems of equations. In addition, a great number of mistakes due to arithmetic difficulties, alone and combined to specifically algebraic difficulties, was detected. The analysis of mistakes in both groups of questions (word problems and equations) suggests, in global terms, an extremely low comprehension of algebra as a representational and solving tool. These data, crossed with observation of class activity in algebra, suggest a very poor pedagogical activity in the introduction to elementary algebra, with a preoccupying failure concerning the construction of meaning and a clear stress on syntactic rules disconnected from their conceptual principles.

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THE MYTHIC QUEST OF THE HERO: A SEMIOTIC ANALYSIS OF MATHEMATICAL PROOF

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A social constructivist account of proof as a means of persuading the mathematical community is now widely accepted. But for those who accept that the traditional absolutist role of logic has been dethroned, a question remains. What does the reader of mathematical proof experience that convinces him or her to accept the theorem, instead of flawless logic? What psychological processes are involved in the reader experiencing the text? This paper offers a tentative social constructivist and semiotic-based theory of proof, which is also intended to describe the way some learners of mathematics interact with texts and tasks.

Every sign, fragment of text, or task in mathematics has two intertwined aspects: that of signifier and signified. My claim is that the realm of signifieds is an imaginary, textually-defined realm, which via processes of intuition ultimately forms the platonic universes that mathematicians' thoughts inhabit. Part of any mathematics learner's or mathematician's role in interpreting a mathematical text is to imagine a miniature math world signified by the text. But in reading a proof or carrying out a classroom task, the reader is following (or accomplishing) the transformation of that text. In doing this, according to Rotman's (1993) analysis the mathematician is carrying out imagined text based actions. In reading a proof, these involve imagined actions coupled with transformations of text which have a cyclic pattern. The beginning is the announcement of the endpoint, the theorem to be proved. This is followed by an imagined voyage through text and underlying math-world, until the endpoint is reached. It is thus a cyclic pattern. According to Rotman (1993) the mathematician alternates his identity or subjectivity between that of the mathematician and his agent: the imagined skeletal representation of self - like the moving fingertip on a map retracing a journey. This representation of self - like the turtle in Logo - traces out a mythic journey of adventure, just as does the universal hero in Campbell's (1956) mythic cycle.

In the paper I explore this analogy, and also that between algorithm and proof, and argue that the latter pair have more in common than is often acknowledged. Thus involvement in the procedures of school mathematics provides an apprenticeship for the future mathematician, in which she learns to project her self into the script, programme, or imagined math-world of the mathematical task. However my conjecture is that the future mathematician learns (1) to obey the imperatives in mathematical text, (2) to write such mathematical texts, and (3) to jump out of the script (i.e. change role from subjected agent to mathematician) and critique it. However many others learn only to be a regulated subject (i.e. 1 above), carrying out on paper and in mind what it needs only a machine to do.

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MATHEMATICS IN THE "AULA-TALLER": IS THERE A BRIDGE CONNECTING THE MATHEMATICS INSIDE AND OUTSIDE THE CLASSROOM?

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The purpose of this study is to develop different alternatives in teaching and learning mathematicas, which take into account the experiences that the students have in their professional enviroment or in the schools where they get a professional training.

These alternatives would deal with how mathematics are used in their context, in their professional enviroment, in their lives, and how this can be incorporated in the academic curriculum. We believe that a deeper understanding of the value given by these students to the mathematics associated to their professional lifes, would help to clarify dark areas in the study of the failure at school. Culture and Mathematics has followed this line of research where the fact that school mathematicas does not take into account the mathematics learnt outside the school is strongly emphasized.

In this research project we follow young people (15 - 19 years old) who dropped the Spanish school system and joined an "Aula Taller" Center, a few years later. This center is located in a suburban area of Madrid, and its purpose is technical education.

We try to explore the different approaches to the learning process in the classroom and the one in the cabinetmaking's workshop, and find out if the way the students experience this interrelation influence their knowledge and beliefs about mathematics. To illustrate both approaches we would like to describe their thinking strategies both in the classroom and in the cabinetmaking's workshop.

We look for the conections between the affective issues and the cultural influences in the mathematical learning. Both generate in the student certain beliefs, and it would be interesting to know to link differences in achievement to beliefs that are connected to cultural influences.

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USING CONCEPT MAPS TO ANALYSE STUDENTS' RELATIONSHIP BETWEEN QUADRILATERALS.

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Concept maps (see Novak et al. 1983) have been used to explore students' understanding in geometry (Mansfield, 1989) and other mathematical topics (see, e.g. Hasemann, 1989) and subjects, mainly in science. We acknowledge, as other authors, that the task of constructing a cognitive map, that is, a concept map made by one student, is dificult and needs previous instruction. But this is not possible if one is involved in a research that explores some topics in a big sample of students. So, it would be interesting if we could design a tool that allowed us to construct cognitive maps from students' answers to a test. With them, one could analyse the relations which a student is capable of establishing and those which he/she is not, and those which are consequence of misconceptions in some of the involved concepts.

Most geometrical concepts are organized in a hierarchical way such as the concepts related to quadrilaterals. This hierarchy can be evidenced throughout propositions involving two concepts and a link that connects both concepts. For example, we can consider the following concepts: Rectangle and Square, and the following links: 'is', 'is not' and 'can be''[1] and try to write some propositions that show what kind of relationship can be constructed between them: 'Square is not a Rectangle' or 'Square is a Rectangle' or Rectangle can be Square' (depending on the definitions used). This information can be brought into a concept map (cognitive map if the relationship are from students) so that we can get a spacial representation of the way that one organizes the concepts in mind.

In the poster we will present how we obtain students' answers about the relationship between quadrilaterals and some cognitive maps constructed from those answers.

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[1] We write 'A is B' because all instances in A are instances in B. In terms of properties, all properties of B are properties of A.
We write 'A can be B' because there are instances in A, but not all, that are instances in B. In terms of properties, some particular properties in A are particular properties in B.
We write 'A is not B' because there are not instances in A that are instance in B. In terms of properties, there are particular properties in A that are not particular properties in B.

ILLUMINATING GAMES PROGRAM FOR KINDERGARTEN MATHEMATICS

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The Illumination-Games Program includes many games and aids that have been especially developed for the mathematics curriculum of the Israeli Ministry of Education.

These illuminating games encompass and teach the mathematical basic concepts. Each of these mathematical concepts appears in a wide range of activities that enables the child to absorb and understand the concepts at his own rate. Moreover, the great variety of mathematical aids and the child's exposure to the educational environment, which our program creates, strengthens the child's command of the concepts and causes him to deepen and internalize these concepts.

This program is built on a modular formation. For example, one of its aids is a set of the "verifier" which has booklets of shapes, patterns, reading numbers and identifying groups, etc. The verifier was described in detail in PME 18 (Orbach & Ilany, 1994). Many of the activities that are suggested for every game are gradually build so that each child can make progress according to his ability. The level of difficulty increases for the different age groups. In the process of playing these games the children can learn the various mathematical concepts in a flexible and easy way. Different children can gain the maximum learning benefits according to their level of development.

In addition, the educational purposes of this program are to teach through enjoyable and creative games, to make children cooperate with each other and to develop proficiency and curiosity. It also gives the opportunity to investigate mathematical situations. Furthermore, it enhances logical thinking as well as encourages the child's creative abilities.

In the conference we will present the games and the aids of our program.

Ref.: Orbach, Haim, & Ilany, Bat-Sheva (1994). Verifier- a self learning method in early geometry and mathematics. PME 18 Conference, (Vol. 1, pp. 99). University of Lisbon, Portugal.

KNOWLEDGE REPRESENTATION - LINEAR SISTEMS

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This paper presents a study with a statistical treatment of identification, interpretation and analysis of the results which appeared in the resolution of linear systems of two equations and two unknown variables, by 10th grade students, using both algebraic and geometric point-of-views.

A projection was made with these results for all the 10th grade population of the city of Rio de Janeiro.

This study relates the representation and the acquisition of a concept which is fundamental in the field of Mahtematical Education.

In the process of teaching-learning, it is essential to have a full consciousness of the stage in wich the student operates in order to render it possible to adapt the teaching methodology to that stage. Before teaching each individual it is necessary to study him/her in order to discover the phase of mental development in which he/she is, as well as his/her specific needs.

Our universe was composed of students in the 10th grade, from 15 to 16 years old.

The chosen theme for this research was "Resolution of Linear Systems of two Equations and two Unknown Variables, under both algebraic and geometric point-of-view". The students should already know this concept at his/her present school grade.

We tried to determine whether this hypothesis was true, and whenever it was not, which difficulties were present and their possible causes.

We have chosen the above subject because it has several uses, both in the scholl perfomance and in practical life.

During the research, five test were applied, three of them with algebraic resolutions and two with geometrical resolution, using grafical representation. These tests were graded, and a statistical treatment was applied to the results, giving the basis for analysing and interpreting the mathematical reasonig of the students, as well as for diagnosing possible causes that might concur for it.

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THE BALANCE SCALE VERSUS PROBLEM-STRUCTURE IN THE INTRODUCTION TO ELEMENTARY ALGEBRA: A COMPARATIVE STUDY

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The two-pan balance scale has been increasingly studied as a pedagogical tool in the introduction to elementary algebra, e.g., Vergnaud & Cortez (1990), Filloy & Rojano (1984), Kobayashi (1988), Schlieman, Santiago & Brito Lima (1990), Da Rocha Falcão (1995) Schlieman, Santiago & Brito Lima (op cit), for example, have shown the importance of the two-pan balance scale as an auxiliary tool for the comprehension of important notions like functional equality and unknown, as well as the main principles involved in algebraic algorithms of equation manipulation. Nevertheless, other researchers, like Both (1967), point out some limitations inherent to the balance-scale metaphor in the introduction to elementary algebra, other authors, like Bondanskii (1990), put more emphasis in the *structure* of the algebraic problems presented to the students as a pedagogical tool, in detriment of concrete metaphors like the balance-scale. This controversy raises an important research question: which aspects, in the *conceptual field* (Vergnaud, 1990) of algebra, are better addressed by auxiliary tools like the two-pan balance-scale, and by a certain problem-structure emphasizing the functional equality between two quantities (to be modeled by an equation). This study tried to offer some elements to this debate; two experimental conditions were then compared: a) a group of 10-11 year-old elementary students (N= 8) was submitted to a set of problem-situations involving the balance-scale; b) another set of students, paired with those from the first set, was submitted to another set of problem-situations, isomorphic to the first set, involving word problems where the equality of quantities were emphasized. Both conditions were presented to the same set of the six following algebraic structures (x and y representing unknowns, other letters representing known quantities):

1 $ax + b = c$	**2** $ax + x = b$	**3** $a + x = bx$
4 $a + x = bx + d$	**5** $a + by = bx + by + d$	**6** $a + by + cx = by + dx + f$

The subjects from the two experimental conditions were previously evaluated with a set of questions (pre-test), involving both problems and equations for each of the six algebraic structures, and post-evaluated with an equivalent set of questions, in order to evaluate eventual differences inter-groups due to experimental design. Results can be summarized as follows: 1. problems involving algebraic structures 1 and 2 were solved by all (100%) of the subjects in the pre-test, with no differences between equations and word problems; 2. Problems involving structure 3, modeled by the equation $x + 36 = 4x$, were interestingly solved in the pre-test by some subjects in the following terms: "*36 must be equal to 3x, because 4x is equal to 3x plus x*"; 3. The global percentage of correct answers in the pre-test was 50%, the other half of the distribution was mainly formed by unsuccessful trial-and-error procedures, giving up problem resolution and algorithm misuse; 4. The percentage of correct answers in the post-test raises to 100% in both equations and word problems, without quantitative difference between the two groups; nevertheless, interesting qualitative problem-solving differences were detected. This last aspect leads to the conclusion that both balance-scale and problem-structure stressing functional equality have pedagogical and probably complementary interest.

TEACHING PROBLEM SOLVING FOR MATHEMATICS TEACHERS

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The Center for Mathematics Education (CEM, São Paulo - Brazil) develops since 1991 a project devoted to training mathematics teachers in their own workplace. The project, centered in Problem Solving, is supported by the Brazilian Federal Government and involves many teachers from several regions of the State of São Paulo.

Several investigations have been conducted in the last few years about the following themes: (1) teachers' beliefs, conceptions and attitude towards mathematics, teaching, and problem solving, (2) problem posing, (3) teacher development.

This poster presents samples of the work developed with 600 teachers who took part in short term courses and with 30 teachers who have been continuously supervised by us. The presentation focuses upon (1) quantitative data from the investigations carried on in the project, (2) the teachers' productions (e.g., journals from daily activities in the classroom), (3) qualitative analyses of several aspects of the investigations, (4) an extensive analysis of the quality of the courses for in-service mathematics teachers.

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A Review in Britain : Epistemology and the Pedagogy of Mathematics

Bea Monteiro – CME/Open University

PME – Recife (Brazil), July 1995

Introduction

Until recently, mathematics and its teaching were seen in the light of logical-positivist traditions, i.e. logical-deductive and therefore absolute, infallible. New alternative epistemological views in the philosophy of mathematics were found to bring consequences for the pedagogy of mathematics.

Review in Britain

Author & main contribution	Description of Study	Alternative View	Method	Conclusion
M. Nickson (1981) Main Contribution Bridge between Curriculum Projects' Views of Mathematics and Practice	(Theoretical – Comparison of 2 main curriculum projects in Britain and analysis of consequence for practice of mathematics within the positivist and growth and change views of mathematics. – Fats to be learned and skills to be mastered. VS – application of facts in problem solving – Inquiry and exploring	Growth & change (Popper, Kuhn, Tulmin, Lakatos)	– Theoretical (qualitative) – Historical review of philosophy of mathematics analysis ideology of curriculum projects	Teachers' change of practice will occur when curriculum project's views of mathematics are based on 'Growth and Change'
S. Lerman (1986) Main Contribution First empirical attempt to connect teachers' views of the nature of mathematics with practice	(Empirical) – Historical review to (a) show the intertwining of absolute knowledge and mathematics and (b) to develop scales to measure teachers' attitudes to mathematical knowledge and compare with practice	Lakatos' Quasi-empiricism	– Empirical (quantitative and qualitative) – Questionnaires (attitude scales) – Classroom observations – Small scale (one school)	Other factors, such as, social factors are also possibly influencing the pedagogy of mathematics
P. Ernest (1991) Main Contribution Development of a social constructivist method of generation of mathematical knowledge	(Theoretical) – Extended Lerman's historical review – Offered critique to quasi-empiricist Lakotosian perspective – Proposed social constructivism	Social Constructivism (cycle of subjective and objective knowledge)	– Extensive historical review – Development of a method of generation of mathematical knowledge using objective and subjective criteria	Mathematics is largely abstract and concepts do not have necessarily origins in the physical world
B. Monteiro (1993) Main Contribution Tested empirically social and cultural factors influencing epistemological views and pedagogy of mathematics	(Empirical) – Followed up Lerman's study and enlarged sample of schools to account for social factors – Teachers' use of computers and software were used as a parameter of teachers' practice	Wittgenstein's (1955) (human logic as not a priori)	– Empirical (quantitative and qualitative) – Questionnaires and interviews in several schools in Oxfordshire	Social and cultural factors also affect teachers' views of the nature of mathematics. New pedagogies may bring changes to views of the nature of knowledge of mathematics.

A DESCRIPTION OF THE STRATEGIES EMPLOYED BY FIRST AND SECOND GRADES STUDENTS IN THE SOLUTION OF MATHEMATICAL PROBLEMS

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The solution of mathematical problems is part of everyone's life in school and daily situations. Different strategies may be used to solve those problems in each of these contexts. In school context mathematics has become outstanding for its high level of unsuccess (Rangel,1992). According to Carraher, Carraher e Schliemann (1993) that happens because algorythms taught in school are an obstacle for children's reasoning. The objective of this study was to compare the way mathematical problems of adition and subtraction are solved by children who are studying the first grade by the second time (i.e. were not successful in the first school year) with the strategies for solution of those problems taught in school. A set of (ten) problems of adition and subtraction (five of each) was presented to two groups of subjects. In Group 1 there were ten students who were studying the first grade by the second time. Group two had ten students of the second grade. The two groups were submitted to the same procedure. Each subject was asked to solve ten mathematical problems (five of adition and five of subtraction). The subject could give either an oral or written response. After the subject presented a solution the experimenter asked about the way through which he/she had reached the solution. Results show the use of algorythmic strategies (AS) and non algorythmic strategies (NAS). Two AS strategies were most frequent: "counting with the fingers" (efficient only in the solution of problems which involve units but not tens) and "decomposition" (the most efficient in the solution of problems which involve units and tens). "Counting with the fingers" was also the most frequent strategy for both groups. And the most efficient strategies for both groups (which most frequently led to correct responses) were decomposition (AS) and "memorization" (NAS).

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[*] CAPES

PROPORTIONALITY: FROM INTUITIVE UNDERSTANDING TO SOLUTION STRATEGIES.

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Previous studies show that, when asked to solve proporcionality problems outside of the specific classes where proportion problem solving is taught, students only rarely attempt to use the rule-of-three algorithm they are taught in schools. Moreover, among the few who try to use the algorithm, most do so as an automatic procedure and, when some part of it is forgotten, they are unable to reconstruct the rule since there is no understanding about the relationships that underlie its steps. Such results suggest that school teaching stressing rules and algorithms learning is setting aside a most important side of mathematics activity: the understanding of relations and the generation of algorithms as a consequence of such relations. However, as shown by Nunes, Schliemann, & Carraher (1993), Magalhaes (1990), Schliemann & Carraher (1992), or Pereira Neto (1992) children and adults with no school instruction on proportionality but who use mathematics in their everyday activities buying and selling goods can solve proportionality problems. They do so by using the scalar strategy which, through a series computations, transforms the values for prices and for measured units preserving, throughout the procedure, the proportional relations between them. Although not as efficient speedwise as the functional strategy or the rule-of-three strategy, the scalar solution is well understood and most frequently leads to correct solutions since it allows for preservation of the meaning of the quantities being dealt with throughout the procedure (see Vergnaud, 1983, for a description of scalar versus functional solutions).

This study aimed at developing more meaningful methods for teaching proportion problem solving in schools by exploring understanding of proportional relations and use of the scalar strategy as a first step.

A total of 25 fifth graders from a public school in Recife, Brazil, participated in the study. Their ages ranged from 11 to 13 years old and they had not received school instruction on proportionality before. Subjects were randomly assigned to three groups, one of them (five subjects) acting as a control group, the other two (total of 20 subjects) constituting the experimental group. The two experimental groups participated in a series of five one-hour teaching sessions.

Teaching sessions for both groups included competitive games and discusssions aimed at creating opportunities for children to reflect upon relationships between series of numbers in order to construct their own solution strategies.

Their development was evaluated through comparison between a pre-test, an immediate post-test and a delayed post test given two months later, as well as through comparisons with the control group. Concerning correct answers, while results for the control group did not show changes accross tests, remaining at around 40% of correct answers throughout, the percentage of correct answers for the experimental group jumped from 57% in the pre-test to 76% in the immediate post-test; in the delayed post-test, however, it decreased to 60%. Statistical comparisons reveal that the difference between number of correct answers in the pre-test and the immediate pos-test was significant ($t19=1,94$, $p=0,05$, unicaudal), but the difference between the pre-test and the delayed post-test was not ($t15=1,67$, $0,05<p<0,10$, unicaudal).

Use of the scalar solution strategy for the experimental group increased from 21% in the pre-test to 44% in the immediate post-test and 57% in the delayed post-test. The functional solution strategy appered in 14% of the problems in the pre-test and in 15% of both post-tests. A notational system developed during the teaching sessions was used to solve 65% of the problems in the immediate post-test. In 26% of the problems the nottion was used but was not the basis for the solution strategy adopted. In the delayed post-test these figures were, respectively, 29% and 57%.

There was therefore a development towards short term use of the scalar strategy with corresponding increase in number of correct answers. Use of the functional strategy, however, remained scarce, probably due to the short number of sessions that did not allow children to explore more advanced solutions.

Representing Decimal Numbers
When Converting Units of Measure

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Research involving students' understanding of decimal numbers has characteristically focused on discontinuities between students' knowledge of natural numbers and knowledge to be acquired regarding rational numbers. For example, whereas multiplication by natural numbers always produces a product larger than the multiplicand, this no longer holds true when the multiplier is a decimal fraction. Similarly, the relative order of two numbers can no longer be reliably determined by considering merely the length of a number, since the shorter decimal number may be larger in value than the longer number.

Decimal numbers are widely used in applied contexts as parts of measures expressed in standard units. This being the case, it is possible that certain problems will arise related not merely to difficulties in understanding the structure of the numerical system but also related to the system of measures and its expression through a numeric system organized according to a different structure. Whereas decimal numbers are expressed according to place value representations structured around powers of ten, the system of measures may entail using varying ratios of units to subunits. For example, in the case of temporal relations, there are 60 minutes in an hour, 24 hours in one day, and so on. Conversion of measures problems are potentially useful in clarifying the difficulties students have in coordinating these two representational systems.

The present research aimed at investigating, during regular school activities and clinical interviews, children's competence with decimal numbers in problems of converting measures along the dimension of time or of distance. Twelve pairs of fifth grade students, aged from 12 to 13 years, were interviewed as they solved three sets of problems involving the following conversions: (1) a time period given in months to be written in years, and vice-versa; (2) a time period given in minutes to be written in hours, and vice-versa, (3) metric distances given in meters to be written as centimeters, and vice-versa. The study also included observations of the subjects' fifth grade classroom. All interviews and classroom observations were videotaped.

The analyses carried out showed that: (1) The teacher's mathematical discourse in the classroom was restricted to tasks of computation and symbolic manipulation of the decimal system and of metric distances, which posed relatively little challenge to students; (2) students frequently understood decimal measures as expressing quantities according to two units of measure (e.g., a major portion of the students understood 2.5 as signifying "2 hours and 5 minutes"), and (3) tasks involving conversions within the metric system tended to be solved through displacements of the decimal point according to rules practiced in school.

This poster will offer a detailed analysis of the strategies employed by the students on the conversion tasks, and discuss instructional approaches to the teaching of decimal numbers with a focus on multiple measuring tasks and activities that can engage students on investigating the relations between numbers and quantities.

Software to Support Reflection, Annotation, and Presentation

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Draft Abstract

Mathematical thinking and learning require frequent access to prior experience for analysis, reflection, and generalization. In addition, organizing personal experiences for public presentation and discussion is necessary both to student learning and authentic assessment. Yet the growing tide of "exploratory" learning tools tends to focus on the continuous push forward to "discovery," and neglects tools for reflection, annotation and presentation.

Within the SimCalc project, a project to build and test a series of rich simulations to support calculus learning beginning in the early grades, we are designing an overall architecture for enabling students to capture their own progress. We want to enable students quickly to revisit previous trials and attach reflections to their work. In addition, we want students to be able exchange their work in e-mail correspondence, or present a slide show for classroom discussion. Furthermore, we are aware that the thrill of discovery often pulls students forward more strongly than the clarity of hindsight draws them back. If students are to use tools for reflection and annotation, the tools must be simple, quick, and not dependent upon compulsive attention to saving work.

The overall metaphor we use is a "dynamic notebook." SimCalc software can save two types of history information: state information, and state change information. Every time the student runs the simulation, a copy of the simulation state is saved to a new page. In addition, student modifications to the simulation between runs can be recorded as commands in the AppleScript language. This recording can occur automatically in the background, thereby preserving state and state change information until students are ready to reflect. We thus overcome the "save as" problem- students don't have remember to save versions of their work, choose names for file versions, etc. Their work automatically is entered into the notebook in natural order.

The easiest way to browse notebook pages is to move forward or backwards in chronological order. Several successive clicks on the next button, for example, produce an flip-chart style animation of the graphs a student produced on successive trials. For non-linear access, students can name pages, and then skip to any page via a popup menu. Within a page, our primary form of annotation is an audio recording of the student's voice. Although we feel students are more likely to speak than to type, we will also support textual notes, as they require less storage space. Notebooks can easily be mailed to a teacher or another student via a Eudora script; furthermore, the ability to flip through successive pages makes a natural format for a full class "slide show," thereby enabling a live performance assessment, as well as portfolio-style assessment.

While the notebook format is still experimental, and subject to refinement after more classroom testing, we believe that it offers a step forward from software applications that offer only "save-as" capability. Coming "component software" technologies such as OpenDoc and OLE may soon make it feasible to provide a generic notebook that could support reflection, annotation and presentation of many kinds of simulation microworld content. We feel that the time is ripe for mathematics educators to design and debate general system capabilities in support of mathematical thinking that could serve as a framework for many diverse "exploratory" software modules. To date, designers of typical systems have concentrated on supporting rich activity and snapshots of the activity in progress or records of completed products. But the support of mathematical growth requires much fuller and more flexible records of activity, records that technological advances make newly possible.

THE INFLUENCE OF REPRESENTATIONS IN THE TEACHING OF MATHEMATICS

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The purpose of this study is to analyse the role of concrete representations, written and mental, in the solution to division problems (partition and quotition). The most of studies have analysed only the role of concrete representations (Desforges & Desforges, 1980; Kouba, 1989). Subjects were 108 children, equally divided across three grades, from kindergarten to second grade. The subject in each series were distibuted in groups which differed with respect to the material made available to assist with the solutions of the problems: group 1 had tokens, group 2 had pencil and paper and group 3 was affered no object. We have observed more correct answers in groups 1 and 2, while the performance of the subjects from these two groups compared presented no significant differences. The main strategies were: direct representations of the problem (the subjects stablished a parallel between their actions and the propositions); trial and error; repeateed adding; use of known fact (multiplication and division). The interaction between strategies of direct representation and repeated adding was significant ($p<.001$ in all grades). Group 1 tended to use a direct representations, while in group 3 repeated addig was more frequent (see figure 1 and 2). These results indicated that the percentage of use of the strategy of representation observed in all grades did not mean lack of knowledge of more sophisticated strategies. It did indicated the influence of the material used on the strategies chosen by the subjects. Similar results were observed in the problems of quotition. Results show that: (a) the importance of written representations produced by children in the development of mathematical concepts and (b) the need to stimulate the use of various types of representations in the mathematical concepts as a way of favouring a broader understanding of the concept being studied.

Figure 1: Percentage of Use of Strategy "Direct Representations with Distribution of Small Quantities" by Group and Grade

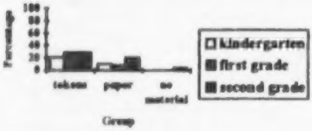
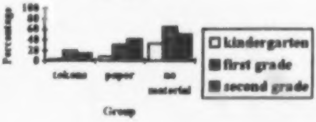


Figure 2: Percentage of Use of Strategy "Repeated Adding" in Problems of Partition by Group and Grade



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QUADRATIC FUNCTIONS AND GRAPHING CALCULATOR

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In this paper we discuss a pedagogical approach for the teaching and learning of quadratic functions using a graphing calculator. We discuss how a high school student deals with quadratic functions using a graphing calculator. Implications for mathematics curriculum are discussed.

Graphing calculators are portable and relatively inexpensive, and can be seen as computers with only one software. They facilitate the production of a graphic representation and allow introduction of an approach which emphasizes visualization.

In this study we worked with the quadratic function written in the form $f(x) = ax^2 + bx + c$. We interviewed one high school student focusing on the effects of the coefficients in the graph.

This is an ongoing study and we are in the process of designing other teaching experiments with another students. We believe that interviewing students allowed us to take a close look at students' reasoning while allowing students to spend the time they need to deal with a given problem.

In this case study, using a graphing calculator, we saw that a student can be provided with tasks that allow him to conduct his own investigations and engage in problems generated by him.

AN EMERGING VISION OF A MATHEMATICS CURRICULUM FOR MIDDLE SCHOOL, AND IMPLICATIONS FOR TEACHER PREPARATION

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The National Center for Research in Mathematical Science Education (NCRMSE), funded by the U.S. Department of Education, has for five years supported a working group which has focused its efforts on research at the middle school level, particularly in the area of multiplicative structures. The working group is directed by Judith Sowder. Two subgroups were formed in the summer of 1994, one to consider the implications of recent research for curriculum at this level, and the other to consider how recent research should affect the preparation of middle school mathematics teachers.

The first subgroup, consisting of Guershon Harel, James Kaput, Richard Lesh, Ricardo Nemirovsky, Randolph A. Philipp, and Patrick Thompson, prepared a document entitled *An Emerging Vision of a Mathematics Curriculum.* This document discusses what is meant by curriculum, and then outlines a conceptual curriculum that focuses on qualities, quantification, reasoning about quantities, thoughtful use of notation, reasoning about representations, and formalizing. Characteristics of good tasks are described, and dilemmas associated with curricula and implementation are discussed.

The second subgroup, in which Barbara Armstrong, Thomas Carpenter, Susan Lamon, Edward Silver, Martin Simon, Larry Sowder, Alba Thompson, and Judith Sowder have participated, prepared a document in which they make recommendations that extend current thinking in teacher preparation to include a deeper understanding of what it means to move from additive to multiplicative reasoning, and how this affects the manner in which teachers must understand quantification and quantitative reasoning, ratio, proportion, and rational number concepts.

A summer 1995 meeting of approximately thirty mathematics educators will focus on reactions to and discussions of these two documents.

The posters at this session will summarize the work of these groups by presenting the recommendations from each of the two groups, and additional recommendations made at the summer meeting.

STUDENTS' INFORMAL AND FORMAL STRATEGIES IN PROPORTION PROBLEM SOLVING

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This exploratory study analyzes the strategies used in proportion problem solving. The analysis of strategies is important because research shows that children use multiple strategies in mathematical problem solving. The strategies are intuitive, related to their previous experience, and informal. However, in formal instruction of proportion problem solving, one strategy, the rule of three algorithm, is the most common, if not only, strategy taught. Considering these two factors and importance of the concept of proportion, it may be helpful to compare the strategies of students with no formal instruction to the strategies of students with formal instruction in proportion problem solving.

One type of proportion is the isomorphism of measures model. The isomorphism of measures models involves two variables: a value on one variable maps to one and only one value on the other variable. Strategies that may be used to solve proportion problems include the scalar strategy, the functional strategy, the unit-rate strategy, and the rule of three algorithm which, as mentioned above, is most commonly the only strategy used in formal instruction.

Two groups of students participate in this study. One group is comprised of three fifth graders who have had no formal instruction in proportion. The other group is comprised of three eighth graders who have had at least two years of formal instruction in proportion. The six students were asked to solve eight word problems in the form of a worksheet. All of the problems were of the isomorphism of measure model and all were missing value problems in which three values are given and one value was unknown. Four of the problems were constructed to be more easily solved by using the scalar strategy, and four by the functional strategy. Students were allowed unlimited time to do the work and allowed to show their work, or just the answer, whichever they preferred. After the student completed the worksheet, she, or he, was interviewed. Each student was asked to explain how the answer for each problem was obtained. Specific questions about strategy were asked in response to the initial explanation and following statements.

The study showed interesting results. First, the fifth graders were able to construct original strategies and were able to use them to solve new types of problems. Secondly, the eighth grade students, who had a very high rate of success, used the scalar and unit-rate strategies to solve approximately half of the problems. The functional strategy was not used and students seemed to ignore or were unable to recognize the relationship of values that related to two different variables. The eighth graders used the rule of three algorithm to solve nearly half of the proportion problems: this result is significant because it differs from previous studies that show less reliance by students to use formal strategies in problem solving. Finally, during the interviews, both groups of fifth graders and eighth graders revealed a high level of metacognition.

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